Fast Derivation Traps

Exercise I. Dirichlet Boundary Conditions.

We consider the shape optimisation minimization problem of the functionnal $J(\Omega)$ of the following form

$$J(\Omega) = j(\Omega, u(\Omega)),$$

on the open sets Ω of $\mathbb{R}^N$, where $u(\Omega)$ is the solution of the heat equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and f is a regular map from $\mathbb{R}^N$ with values in $\mathbb{R}$.

1. Computation of the differential in a simple case.

We seek to determine the differential of u with respect to Ω in a very simple case. We consider the case where $\Omega =]L_1, L_2[$ and $f = 1$. Compute explicitly the solution of the heat equation as well as its eulerian differential along a direction $\theta : \mathbb{R} \to \mathbb{R}$. We recall that the eulerian differential $u'(\Omega)$ in the dorection θ is the map $u'(\Omega) : \Omega \to \mathbb{R}$ defined for all $x \in \Omega$ by

$$u'(\Omega)(x) = \lim_{t\to 0} \frac{u(\Omega_t)(x) - u(\Omega)(x)}{t}, \qquad (1)$$

where $\Omega_t = X(\Omega, t)$ and $X : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{cases} \dot{X}(t, x) = \theta(X(t, x) \\ X(t = 0, x) = x, \end{cases}$$

$\dot{X}$ being the differential of X with respect to t. Note that (1) is correctly defined as for all $x \in \Omega$, $x \in \Omega_t$ for t small enough.

2. We consider tha "natural" Lagrangien associated to the minimization problem J with respect to Ω, that is

$$\mathcal{L}(\Omega, u, p) = j(\Omega, u) + \int_\Omega \nabla u \cdot \nabla p - fp\, dx.$$

In particular for all $p \in H^1_0(\Omega)$, we have

$$J(\Omega) = \mathcal{L}(\Omega, u(\Omega), p). \qquad (2)$$

a. Can the equation (2) be differentiated with respect to Ω if p is independant of Ω ?

b. Let $p(\Omega)$ be a function that maps Ω to an element of $H^1_0(\Omega)$. For all such functions, we have

$$J(\Omega) = \mathcal{L}(\Omega, u(\Omega), p(\Omega)). \qquad (3)$$

Assuming that both $p(\Omega)$ and $u(\Omega)$ to e both differentiable, differentiate equation (3) with respect to Ω.

c. Compute the partial differential of $\mathcal{L}$ with respect to Ω, u et p.

d. Prove that there exosts a unique $p(\Omega) \in H^1_0(\Omega)$ such that

$$\langle \frac{\partial \mathcal{L}}{\partial u}, q\rangle = 0, \qquad (4)$$

for all $q \in H^1_0(\Omega)$. Check that for all $v \in H^1_0(\Omega)$, we have

$$\langle \frac{\partial \mathcal{L}}{\partial p}, v\rangle = 0. \qquad (5)$$

e. Can an expression of $J'(\Omega)$ depending only on Ω, $u(\Omega)$ and $p(\Omega)$ be obtained from (3), (4) and (5) ?

3. We recast the heat equation as follows: Find $u \in H^1(\Omega)$ and $\lambda \in H^{-1/2}(\partial\Omega)$ (you don't really have to wonder what $H^{-1/2}(\partial\Omega)$ is; let's just say that it is the image of $H^1(\Omega)$ under the trace operator), such that for all $p \in H^1(\Omega)$ and all $\hat{\lambda} \in H^{-1/2}(\partial\Omega)$, we have

$$\int_\Omega \nabla u \cdot \nabla p\, dx = \int_{\partial\Omega} \lambda p\, dx + \int_\Omega fp\, dx, \qquad (6)$$

and

$$\int_{\partial\Omega} \hat{\lambda} u\, dx = 0. \qquad (7)$$

a. Assuming that the solution of the heat equation is regular prove that it indeed satifies (6) et (7). Express λ in function of u.

b. Introduce the Lagrangian $\mathcal{L}_D$ associated to the minimization problem of $j(\omega, u)$ under the constraints (6) and (7). Prove that there exists a couple

$p(\Omega) \in H^1(\Omega)$ and $\hat{\lambda}(\Omega) \in H^{-1/2}(\partial\Omega)$ such that

$$\frac{\partial \mathcal{L}_D}{\partial u}(\Omega, u(\Omega), \lambda(\Omega), p(\Omega), \hat{\lambda}(\Omega)) = 0$$

and

$$\frac{\partial \mathcal{L}_D}{\partial \lambda}(\Omega, u(\Omega), \lambda(\Omega), p(\Omega), \hat{\lambda}(\Omega)) = 0.$$

c. Determine an expression of $J'(\Omega)$ only dependint on Ω, $u(\Omega)$ and $p(\Omega)$. Is the expression obtained different from the one obtained using a "naïve" applicaation of the fast differential method ?

Exercise II. Jumps of conductivity.

We consider the shape optimization problem consisting to determine the best repartition of two different material of respective conductivity A_0 and A_1 (A_0 and A_1 are definite symetric matrices). The solid Ω is made of a matrix of conductivity A_1 and contains an inclusion Ω_0 of conductivity A_0 (we assume $\overline{\Omega}_0 \subset \Omega_1$). Our optimization parameter is the inclusion Ω_0. We deonte by $u(\Omega_0) \in H^1_0(\Omega)$ the solution of the heat equation, such that for all $v \in H^1_0(\Omega)$, we have

$$\int_{\Omega_0} A_0 \nabla u(\Omega_0) \cdot \nabla v \, dx + \int_{\Omega_1} A_1 \nabla u(\Omega_0) \cdot \nabla v \, dx = \int_{\Omega} f v \, dx,$$

where f is a regular map from Ω into $\mathbb{R}$ and $\Omega_1 = \Omega \setminus \Omega_0$. We seek for an expression of the shape derivative of

$$J(\Omega_0) = j(\Omega_0, u(\Omega_0)),$$

where j is a regular map.

1. We introduce the "natural" Lagrangian

$$\mathcal{L}(\Omega_0, u, p) = j(\Omega_0, u) + \int_{\Omega_0} A_0 \nabla u(\Omega_0) \cdot \nabla p \, dx + \int_{\Omega_1} A_1 \nabla u(\Omega_0) \cdot \nabla p \, dx - \int_{\Omega} f p \, dx,$$

so that for all $p \in H^1_0(\Omega)$, we have

$$J(\Omega_0) = \mathcal{L}(\Omega_0, u(\Omega_0), p). \quad (8)$$

a. Prove that there exists a unique $p(\Omega_0) \in H^1_0(\Omega)$ such that for all $q \in H^1_0(\Omega)$, we have

$$\langle \frac{\partial \mathcal{L}}{\partial u}(\Omega_0, u(\Omega_0), p(\Omega_0)), q \rangle = 0.$$

b. Can an expression of $J'(\Omega_0)$ depending only on Ω_0, $u(\Omega_0)$ and $p(\Omega_0)$ be obtained using the introduced Lagrangian ?

2. We recast the heat equation problem as follows: Find

$$u_1 \in V := \{v \in H^1(\Omega_1) \text{ such that } v = 0 \text{ on } \partial\Omega\},$$

$u_0 \in H^1(\Omega_0)$ and $\lambda(\Omega_0) \in H^{-1/2}(\partial\Omega_0)$ such that for all $p_1 \in V$, all $p_0 \in H^1(\Omega_0)$ and all $\hat{\lambda} \in H^{-1/2}(\partial\Omega_0)$, we have

$$\int_{\Omega_0} A_0 \nabla u_0(\Omega_0) \cdot \nabla p_0 - f p_0 \, dx + \int_{\Omega_1} A_1 \nabla u_1(\Omega_0) \cdot \nabla p_1 - f p_1 \, dx + \int_{\partial\Omega_0} (p_0 - p_1) \lambda \, ds = 0, \quad (9)$$

and

$$\int_{\partial\Omega_0} (u_0 - u_1) \hat{\lambda} \, ds = 0. \quad (10)$$

a. Intorduce the Lagrangian associated to this minimization problem of $j(\Omega_0, u)$ under the constriants (9) and (10).

b. Compute the differential of J with respect to Ω_0 by the mean of the fast differentiation method.