Geometric Optimization

Exercise I. Optimization of the rigidity of an elastic structure.

We want to optimize the shape of an elastic body in order to maximize its rigidity.

Let Ω be a bounded connected open set in $\mathbb{R}^N$. We assume that the body is clamped on Γ_D and submitted to surface loads g on Γ_N while it is left free on Γ, with $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$. Moreover, we assume that it is submitted to volumic loads $f \in L^2(\mathbb{R}^N)$. We only optimize the part Γ of the boundary of Ω.

Our aim is to minimize the compliance of the structure, that is

$$J(\Omega) = \int_\Omega f \cdot u dx + \int_{\Gamma_N} g \cdot u ds,$$

where u stands for the displacement of the structure at the equilibrium state.

1. Recall the variational problem verified by the displacement $u(\Omega)$ of the structure.
2. Introduce the Lagrangian $\mathcal{L}(\Omega, u, p)$ associated to the minimization problem considered, where $u, p \in H^1(\mathbb{R}^N)$, $u = p = 0$ on Γ_D and $\Gamma_D \cup \Gamma_N \subset \partial\Omega$.
3. Compute the derivatives of $\mathcal{L}$ with respect to u and Ω. Assuming that u is itself differentiable with respect to the domain, deduce an expression of $J'(\Omega)$.
4. In the case $f = 0$, prove that is it always interesting to add some material to the shape in order to minimize its compliance. If a constraint is added on the total volume of the structure, that is if the admissible shapes is defined as the one such one

$$\int_\Omega 1 dx \leq V_0,$$

give the optimality conditions (of order one) of the problem.

Exercise II. Optimization of a heating.

We consider the optimization of a heater in a room Ω. The shape of the heater is denoted $\omega \subset \Omega$. We assume that the temperature outside is null outside the domain Ω and equal to a given constant $T_1 > 0$ on the heater. Moreover, the room is filled by an air stream of velocity u assumed to be divergence free. The temperature T in the room verifies the equation

$$\begin{cases} -\Delta T + u \cdot \nabla T = 0 & \text{in } \Omega \setminus \omega \\ T = 0 & \text{on } \partial\Omega \\ T = T_1 & \text{on } \partial\omega. \end{cases} \quad (1)$$

Our aim is to obtain a homogeneous temperature T_0 in the room. To this end, we introduce the cost function

$$J(\omega) = \int_{\Omega - \omega} |T(\omega) - T_0|^2 dx,$$

where $T(\Omega)$ is the solution of (1).
1. Determine the variational formulation associated to (1). Prove that is admits a unique solution.
2. Recast the variational problem (1) expressing the constraint $T(\omega) = T_1$ on $\partial\omega$ by introducing a Lagrange multiplier $\lambda(\omega)$.
3. Recast the minimization problem of J as a constrained minimization problem for which the evaluation of the cost function do not rely on the resolution of a variational problem.
4. Determine the Lagrangian associated to the minimization problem introduced. Assuming that $T(\omega)$ and $\lambda(\omega)$ to be both differentiable with respect to ω, compute the geometric differential $\langle \partial J/\partial\omega, \theta \rangle$ for all fields θ such that $\theta \cdot n = 0$ on $\partial\Omega$.

Exercise III. Optimization of a grip.

We want to optimize the shape of a grip with no joint. Let Ω be the shape of the grip. The grip is submitted to loads f on a part Γ_N of its boundary. An object (assumed to be undeformable) is placed under its claws. We suppose that the grip is clamped on a part Γ_D of its boundary and free on the remaining denoted Γ. We want to maximize the grip on the object, that is its pressure, that is

$$P(\Omega) = -\int_{\Gamma_G} (\sigma(\Omega)n) \cdot n \, ds,$$

where

$$\sigma(\Omega) = 2\mu e(u) + \lambda(\operatorname{div} u)\,\mathrm{Id}$$

is the tensor of constraints. Finally, the grip is assumed to be made of a linear elastic material. thus, the displacement u of the grip is the solution of the variational problem.

$$u \in V := \{v \in H^1(\Omega)^n \;:\; (v \cdot n) = 0 \text{ on } \Gamma_G \text{ and } u = 0 \text{ on } \Gamma_D\},$$

for all $v \in V$,

$$\int_\Omega 2\mu e(u) \cdot e(v) + \lambda(\operatorname{div} u)(\operatorname{div} v) dx = \int_{\Gamma_N} f \cdot v \, ds.$$

1. Prove that if $u_c \in H^1(\Omega)$ is such that $u_c = 0$ on Γ_N and $u_c = n$ on $\Gamma_G \cup \Gamma_D$, then $P(\Omega) = -J(\Omega)$, where

$$J(\Omega) = \int_\Omega 2\mu e(u) \cdot e(u_c) + \lambda(\operatorname{div} u)(\operatorname{div} u_c) dx.$$

2. We want to minimize J solely by optimizing part Γ of the boundary $\partial\Omega$. Determine the geometrical differential of J along vector fields that are null on $\Gamma_N \cup \Gamma_G \cup \Gamma_D$. Define the associated Lagrangian and adjoint state to this end.

Exercise IV. Maximization of the smallest eigenvalue.

We want to optimize the shape of a membrane so to maximize the associated smallest eigenvalue of the Laplacian with Dirichlet boundary conditions.

1. Introduce the Lagrangian $\mathcal{L}(\Omega, u, \lambda, p, \mu)$ associated to the minimization problem, where u, $p \in H^1(\mathbb{R}^N)$, $\lambda \in \mathbb{R}$, and $\mu \in H^1(\mathbb{R}^N)$ (The boundary Dirichlet conditions verified by u are introduced be the mean of a new constraint that associated Lagrangian multiplier is μ).

2. Derive the first order optimality conditions under a constraint of constant volume V_0.

3. Does an optimal shape of the maximization problem with constraint volume exists ? Are the previous optimality conditions sufficient ?

Exercise V. Minimization of the vibrations of a membrane.

We consider a new optimization problem, of the same type than the previous case, where the cost function depends not on the smallest eigenvalue but the corresponding eigen-vector u.

Let Ω be a regular connected open set of $\mathbb{R}^N$. We introduce the cost function

$$J(\Omega) = \frac{\int_\Omega u^2 dx}{\left(\int_\Omega u dx\right)^2},$$

where u is the eigen-vector associated to the smallest eigen-value of the Laplacian with boundary conditions. We recall that u is of constant sign on Ω so that the denominator of the fraction which defines J is never null.

1. Prove that J is independent of the chosen eigen-vector associated to λ.

2. Compute the geometric differential of J with respect to Ω.

3. Prove that the adjoint problem introduced admits a unique solution in $H^1(\Omega)$.

Exorcise VI. From parametric to geometric optimization.

We consider the minimization problem of the function

$$J(D) = \int_\Omega |u(D) - u_0|^2 dx,$$

with respect to $D \in L^\infty(\Omega; \mathbb{R})$, where $u_0 \in L^2(\Omega)$, Ω is an open bounded regular set of $\mathbb{R}^N$ and $u(D)$ is the solution of the limit problem

$$\begin{cases} -\nabla \cdot (D\nabla u) = 0 & \text{in } \Omega, \\ D\nabla u \cdot n = g & \text{on } \Gamma_N, \\ u = 0 \text{ on } \Gamma_D, \end{cases} \tag{2}$$

with $g \in L^2(\Gamma_N)$ and $\partial\Omega = \Gamma_N \cup \Gamma_D$,such that Γ_D and Γ_N are of non zero measure.

1. Let D such that $D \geq D_0$ a.e. where D_0 is a positive real.

a. Determine the variational formulation associated to the limit problem (2).
b. Prove that the variational problem obtained admits a unique solution.
2. Prove that $u(\cdot) : L^{\infty}(\Omega) \to H^1(\Omega)$ is continuous and derivable for all $D \geq D_0 > 0$. Deduce an expression of the differential of J.
3.a. Restate the minimization problem of J as a constraint minimization problem whose evaluation do not require the resolution of a variational problem.
b. Introduce the associated Lagrangian.
c. Deduce a new expression of the differential of J. What is its advantage with the expression obtained by the previous question ?
4. We introduce a new notion of differentiation. For all vector fields $\theta \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ such that $\theta \cdot n = 0$ on $\partial\Omega$ we define the map $X_\theta : \Omega \times]-\infty, \infty[\to \Omega$

$$\begin{cases} \frac{\partial X_\theta}{\partial t}(x,t) = \theta(X_\theta(x,t)) & \text{for all } (x,t) \in \Omega \times \mathbb{R} \\ X_\theta(x,0) = x & \text{for all } x \in \Omega. \end{cases}$$

In other words, $X_\theta(x,\cdot)$ describes the trajectory of a particle placed at x at $t=0$ with velocity θ. For all time t, $X_\theta(\cdot,t)$ defines a diffeomorphism from Ω to Ω. For all elements D in $L^\infty(\Omega)$, we denoted $D_\theta(t)$ the element of $L^\infty(\Omega)$ defined by

$$D_\theta(t)(X(x,t)) = D(x) \text{ for all } x \in \Omega.$$

a. We assume D to be regular. Prove that $D_\theta(t)$ is derivable with respect to t at $t=0$ and compute its differential.
b. Prove that the map $\theta \mapsto J(D_\theta(t))$ is derivable at $t=0$ for all regular D such that $D \geq D_0 > 0$. Determine its differential denoted $\langle \partial J/\partial\Omega, \theta\rangle$.
5. We would like to apply this new notion of derivation for diffusion coefficients D constant by parts. To this end, we propose to formally pass to the limit in the expression obtained in the continuous case. Let ω be an open regular set included in Ω. Let $D \in L^\infty(\Omega;\mathbb{R}^N)$ defined for all $x \in \Omega$ by

$$D(x) = \begin{cases} D_0 & \text{if } x \notin \omega \\ D_1 & \text{if } x \in \omega. \end{cases}$$

We assume that there exists a sequence of regular maps D_n converging toward D.

a. Let n be the outward normal to ω and τ the tangent vector to the boundary of ω. In the following we assume that (τ, n) can be extended on the whole set Ω by a regular filed such that $(\tau(x), n(x))$ is a base for all x in Ω. Prove that for all regular fields σ_n such that σ_n converges toward σ in $H^1(\Omega)^2$,

$$\int_\Omega \nabla D_n \cdot \sigma_n dx \to \int_{\partial\omega} (D_0 - D_1)\sigma \cdot n ds.$$

b. We denote u_n and p_n the solutions of the state and adjoint equations associated to D_n. We assume that

$$\frac{\partial u_n}{\partial \tau} \to \frac{\partial u}{\partial \tau} \qquad \text{and} \qquad D_n \frac{\partial u_n}{\partial n} \to D\frac{\partial u}{\partial n}.$$

in $H^1(\Omega)$. Determine the expression $\langle \partial J/\partial\Omega(D), \theta\rangle$ by computing the limit of $\langle \partial J/\partial\Omega(D_n), \theta\rangle$.