Mathematical Models and Methods in Applied Sciences
Vol. 20, No. 8 (2010) 1251–1297

DOI: 10.1142/S0218202510004593

MATHEMATICAL ANALYSIS OF A SAINT-VENANT MODEL WITH VARIABLE TEMPERATURE

VINCENT GIOVANGIGLI

CMAP, CNRS, École Polytechnique, 91128 Palaiseau Cedex, France

BINH TRAN

Saint-Gobain Recherche, 39 Quai Lucien Lefranc,
93303 Aubervilliers Cedex, France

Received 20 June 2009
Revised 20 October 2009
Communicated by E. Sanchez-Palencia

We investigate the derivation and the mathematical properties of a Saint-Venant model with an energy equation and with temperature-dependent transport coefficients. These equations model shallow water flows as well as thin viscous sheets over fluid substrates like oil slicks, atlantic waters in the Strait of Gilbraltar or float glasses. We exhibit an entropy function for the system of partial differential equations and by using the corresponding entropic variable, we derive a symmetric conservative formulation of the system. The symmetrized Saint-Venant quasilinear system of partial differential equations is then shown to satisfy the nullspace invariance property and is recast into a normal form. Upon establishing the local dissipative structure of the linearized normal form, global existence results and asymptotic stability of equilibrium states are obtained. We finally derive the Saint-Venant equations with an energy equation taking into account the temperature-dependence of transport coefficients from an asymptotic limit of a three-dimensional model.

Keywords: Saint-Venant model; shallow water model; thin viscous sheet; fluid substrate; free boundary.

AMS Subject Classification: 35Q35, 76A20, 76D27

1. Introduction

We investigate the derivation and mathematical properties of a viscous Saint-Venant system of partial differential equations with an energy equation and with temperature-dependent transport coefficients. These equations model shallow water flows as well as thin viscous sheets over fluid substrates like oil slicks on water, surface atlantic waters above the denser Mediterranean sea in the Strait of Gilbraltar or float glasses used for the production of plate glass. Modeling temperature variations is important in various environmental and engineering applications like float glasses and this motivates the present study.

We first present the Saint-Venant system of partial differential equations with an energy equation and temperature-dependent transport coefficients. We exhibit an entropy function for the system of partial differential equations and by using the corresponding entropic variable, we derive a symmetric conservative formulation of the system. The symmetrizing variable is obtained from the entropy and not from the kinetic energy as investigated by Tadmor,[55] Hauke,[29] and Carey[12] in the isothermal case. These symmetrized systems may also be useful for finite element discretizations and numerical simulations as investigated by Hughes, Franca and Mallet,[33] Chalot, Hughes and Shakib,[13] Hauke[29] and Carey.[12]

The symmetrized Saint-Venant system of partial differential equations is then shown to satisfy the nullspace invariance property and is recast into a normal form, that is, in the form of a symmetric hyperbolic−parabolic composite system. We next establish stability conditions of the source term as well as the local dissipative structure of the linearized normal system around constant equilibrium states. In particular, the entropy production is non-negative and the source term lies in the range of its derivative at equilibrium. Global existence results and asymptotic stability of equilibrium states are then obtained from Kawashima's theory of hyperbolic−parabolic systems[37] and its extension to systems with source terms.[25]

Numerous existence results can be found in the literature concerning the Saint-Venant system without an energy equation in various functional settings. We refer the reader notably to Serre,[52] Dafermos,[14] Sanchez-Hubert and Sanchez-Palencia[51] for inviscid models, and Kanayama and Ushijima,[35] Bernardi and Pironneau,[5] Ton,[56] Kloeden,[39] Sundbye,[54] Orenga,[45] Lions,[42] and Wang and Xu[59] for viscous Saint-Venant models with constant viscosity coefficients. Global weak solutions have also been investigated by Bresch,[8] Bresch and Desjardins,[9] Bresch, Desjardins and Métivier,[10] and Li *et al.*[41] with density dependent viscosities, using a gradient entropy,[8] and Li *et al.*[41] also considered the vanishing of vacuum states. Initial value problems have also been studied with various natural boundary conditions and we refer to Sanchez-Hubert and Sanchez-Palencia,[51] Sundbye,[54] Orenga,[45] Bresch,[8] Levermore and Sammarino,[40] and Li *et al.*[41] Note that, when there are vacuum states at the boundary, the boundary conditions disappear.[8,41,51] To the authors' knowledge, it is the first time that the quasilinear Saint-Venant model with an energy equation and temperature-dependent transport coefficients is investigated.

In the remaining part of the paper we derive the Saint-Venant equations with an energy equation taking into account the temperature-dependence of transport coefficients. These equations are derived from an asymptotic study of a three-dimensional incompressible model of a thin viscous sheet over a fluid substrate. The fluid substrate is incompressible and is modeled by using the hydrostatic approximation. We also derive typical free boundary conditions for the Saint-Venant model from the three-dimensional governing equations and free boundary conditions associated with the viscous sheet.

Numerous derivations of the viscous Saint-Venant system of partial differential equations without an energy equation and with a constant viscosity can also be found in the literature. The inviscid equations were first written by Saint-Venant in 1871.[16] The viscous equations have been investigated by Kanayama and Ushijima[34] and Gerbeau and Perthame[19] who further validated the Saint-Venant model by a direct numerical comparison with the underlying incompressible model.[19] Bresch and Noble also investigated the mathematical derivation of shallow water type equations with non-flat bottoms. For viscous layers on a fluid substrate, Howell has derived a Saint-Venant model by performing an asymptotic analysis.[31,32] Multilayer Saint-Venant models have recently been investigated by Audusse,[1] Audusse and Bristeau,[2] and Kanayama and Dan.[36] A Saint-Venant model with a temperature equation has been introduced by Benqué, Haugel, and Viollet[3] and used by Podsetchine, Schernewski, and Tejakusuma[48] to investigate the Oder Lagoon. The derivation of a Saint-Venant model of a thin viscous sheet over a fluid substrate with a temperature equation and taking into account the temperature-dependence of transport coefficients as well as that of boundary conditions from an asymptotic analysis is new to the authors' knowledge.

2. Governing Equations

We summarize in this section the Saint-Venant equations governing thin viscous sheets over fluid substrates as well as shallow water flows. We include an energy equation in the model since temperature variations are important in various engineering and environmental applications.

2.1. *Conservation equations*

The equations governing shallow water flows and thin viscous sheets over fluid substrates express the conservation of mass, horizontal momentum and energy. The mass conservation equation can be written in the form

$$\partial_t h + \partial_x(hu) + \partial_y(hv) = 0, \tag{2.1}$$

where t denotes time, (x, y) the horizontal Cartesian coordinates, h the height of the shallow water flow or of the viscous sheet in the vertical direction, u the velocity in the x-direction, and v the velocity in the y-direction. The momentum equations in the x- and y-directions can be written as

$$\partial_t(hu) + \partial_x(hu^2 + p) + \partial_y(huv) + \partial_x \Pi_{xx} + \partial_y \Pi_{xy} = 0, \tag{2.2}$$

$$\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + p) + \partial_x \Pi_{yx} + \partial_y \Pi_{yy} = 0, \tag{2.3}$$

where p is the kinematic pressure and Π_{xx}, Π_{xy}, Π_{yx}, and Π_{yy} are the coefficients of the kinematic viscous tensor $\boldsymbol{\Pi}$. Finally the total energy conservation equation can be written in the form

$$\begin{aligned}\partial_t(he^{\mathrm{tot}}) &+ \partial_x((he^{\mathrm{tot}} + p)u) + \partial_y((he^{\mathrm{tot}} + p)v) \\ &+ \partial_x(\mathcal{Q}_x + \Pi_{xx}u + \Pi_{xy}v) + \partial_y(\mathcal{Q}_y + \Pi_{yx}u + \Pi_{yy}v) = \mathcal{H},\end{aligned} \tag{2.4}$$

where e^{tot} is the total energy per unit mass, $\mathcal{Q}_x$, $\mathcal{Q}_y$ are the components of the kinematic heat flux $\boldsymbol{Q}$ and $\mathcal{H}$ denotes the heat loss term.

Since the Saint-Venant system of partial differential equations is naturally written in two dimensions, we will use in the following sections the indexing set $C = \{x, y\}$ which is more explicit than the set $C = \{1, 2\}$.

2.2. *Thermodynamic properties*

In the Saint-Venant system, the kinematic pressure is given by

$$p = \frac{1}{2}\alpha h^2, \tag{2.5}$$

where α is a constant associated with gravity. On the other hand, the total energy per unit mass e^{tot} of the fluid sheet is given by

$$e^{\mathrm{tot}}(h, T) = e + \frac{1}{2}(u^2 + v^2), \tag{2.6}$$

where e denotes the fluid sheet internal energy per unit mass. The internal energy e can be written as

$$e(h, T) = e^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_v(\tau)\, d\tau + \frac{1}{2}\alpha h, \tag{2.7}$$

where c_v is the heat capacity at constant volume per unit mass of the fluid, T the absolute temperature and e^{st} the formation energy of the fluid at the standard temperature T^{st}. We also define for convenience the formation energy at zero temperature $e^0 = e^{\mathrm{st}} - \int_0^{T^{\mathrm{st}}} c_v(\tau)\, d\tau$ in such a way that the internal energy e can also be written as $e = e^0 + \int_0^T c_v(\tau)\, d\tau + \frac{1}{2}\alpha h$.

In comparison with the perfect gas model, we note that, with the Saint-Venant system modeling fluid sheets, the height h plays the role of a density, the fluid is barotropic with a quadratic dependence of the pressure p on height h and the internal energy per unit mass of the fluid sheet e depends on both temperature T and height h.

The natural compatibility relation[42] between p and e is also satisfied since $h^2\partial_h e = p - T\partial_T p = \frac{1}{2}\alpha h^2$ so that there exists an entropy per unit mass s such that Gibbs relation $T\, ds = de + pd(1/h)$ holds. From Gibbs relation, it is easily shown that $T\partial_T s = \partial_T e = c_v$ and $T\partial_h s = \partial_h e - p/h^2 = 0$ in such a way that

$$s = s^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} \frac{c_v(\tau)}{\tau}\, d\tau, \tag{2.8}$$

where s^{st} is the formation entropy of the fluid at temperature T^{st}. The Gibbs function is further defined as $g = e + p/h - Ts$ and will be required to express the entropic symmetrizing variable. Note finally that the Gibbs function g can be decomposed into $g(h, T) = \bar{g}(T) + \alpha h$ where $\bar{g}$ only depends on temperature and reads $\bar{g} = e^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_v(\tau)\, d\tau - T(s^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} \frac{c_v(\tau)}{\tau}\, d\tau)$.

Remark 2.1. Strictly speaking, denoting by ρ the — constant — density of the fluid, only the quantity $\rho p/h$ is homogeneous to a pressure and p/h to a kinematic pressure. However, these h factors are natural since the equations are in two dimensions so that the internal constraints are transmitted through contact lines and not contact surfaces. Similarly, the quantity ρeh is the internal energy per unit horizontal surface and ρsh the entropy per unit horizontal surface of the fluid sheet.

2.3. *Transport fluxes*

The transport fluxes of the fluid sheet, that is, the kinematic viscous tensor $\boldsymbol{\Pi}$ and the kinematic heat flux $\boldsymbol{Q}$, can be obtained from an asymptotic analysis as presented in Sec. 7. The kinematic viscous tensor is of the form

$$\boldsymbol{\Pi} = -\nu h(\boldsymbol{\partial_x v} + \boldsymbol{\partial_x v}^t + 2\boldsymbol{\partial_x} \cdot \boldsymbol{vI}), \tag{2.9}$$

where $\boldsymbol{\partial_x}$ denotes the derivation vector $\boldsymbol{\partial_x} = (\partial_x, \partial_y)^t$, $\boldsymbol{v}$ the velocity vector $\boldsymbol{v} = (u, v)^t$, $\boldsymbol{x}$ the component $\boldsymbol{x} = (x, y)^t$, ν the kinematic shear viscosity of the fluid, $\boldsymbol{I}$ the two-dimensional unit tensor, and superscript t indicates the transposition operator. The viscous tensor $\boldsymbol{\Pi}$ thus corresponds to the usual two-dimensional formulation with a "shear viscosity" $h\nu$ and a "volume viscosity" $3h\nu$. There is thus a volume viscosity term as for polyatomic gases.[7] Upon decomposing the viscous tensor, we obtain

$$\boldsymbol{\Pi} = \begin{pmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{yx} & \Pi_{yy} \end{pmatrix} = -\nu h \begin{pmatrix} 2(2\partial_x u + \partial_y v) & \partial_y u + \partial_x v \\ \partial_y u + \partial_x v & 2(\partial_x u + 2\partial_y v) \end{pmatrix}. \tag{2.10}$$

We also define, for future use, the kinematic pressure tensor $\boldsymbol{P} = p\boldsymbol{I} + \boldsymbol{\Pi}$, which can be interpreted as a kinematic momentum flux tensor. In addition, the kinematic heat flux is given by

$$\boldsymbol{Q} = (\mathcal{Q}_x, \mathcal{Q}_y)^t = -\varkappa h \boldsymbol{\partial_x} T, \tag{2.11}$$

where $\varkappa$ is the kinematic thermal conductivity of the fluid.

Remark 2.2. Strictly speaking, denoting by ρ the — constant — density of the fluid, only the quantity $\rho\boldsymbol{\Pi}/h$ is homogeneous to a viscous tensor and $\boldsymbol{\Pi}/h$ to a kinematic viscous tensor. Similarly, only the quantity $\rho\boldsymbol{Q}/h$ is homogeneous to a heat flux and $\boldsymbol{Q}/h$ to a kinematic heat flux. However, these h factors are natural since the internal constraints are transmitted through contact lines in two-dimensional models. We still denote $\boldsymbol{\Pi}$ the "viscous tensor" and $\boldsymbol{Q}$ the "heat flux" for the sake of simplicity. The quantities $\eta = \nu\rho$ and $\lambda = \varkappa\rho$ are the dynamic viscosity and the thermal conductivity, respectively, of the fluid.

Remark 2.3. *Erroneous* forms of the viscous terms are often found in the literature as for instance the forms $-\boldsymbol{\partial_x} \cdot (\nu(\boldsymbol{\partial_x}(h\boldsymbol{v}) + \boldsymbol{\partial_x}(h\boldsymbol{v})^t + 2\boldsymbol{\partial_x} \cdot (h\boldsymbol{v})\boldsymbol{I}))$ or $-h\boldsymbol{\partial_x} \cdot (\nu(\boldsymbol{\partial_x v} + \boldsymbol{\partial_x v}^t + 2\boldsymbol{\partial_x} \cdot \boldsymbol{vI}))$ instead of the correct form obtained from asymptotics $-\boldsymbol{\partial_x} \cdot (\nu h(\boldsymbol{\partial_x v} + \boldsymbol{\partial_x v}^t + 2\boldsymbol{\partial_x} \cdot \boldsymbol{vI}))$. Only the correct later form is energetically consistent as shown by Gent.[18]

2.4. *Source terms*

Heat exchanges are important in the modeling of shallow water flows[48] and various viscous sheets such that oil slicks and float glasses.[31] We consider a heat loss term in the form

$$\mathcal{H} = -\lambda^*(T - T^{\mathrm{e}}),$$

where T^{e} is a given constant ambiant temperature and λ^* a heat exchange coefficent.

Remark 2.4. Various other effects may be taken into account in the Saint-Venant system of partial differential equations depending on the particular application under consideration. For shallow water flows, it is possible for instance to take into account friction forces, wind effects, coriolis forces due to earth rotation and the sea depth.[12,19] In the modeling of oil spills it is also important to take into account friction forces, water currents, shoreline deposition, wind effects and evaporation.[46] These extra source terms would not essentially modify the mathematical analysis that will be presented in the following sections.

Remark 2.5. Depending on the particular application under investigation, various terms may also be neglected in the Saint-Venant system of partial differential equations as for instance the kinetic energy terms in the energy conservation equation. However, these terms have been kept since they are important for structural purposes. They guarantee that the structure of the system is that of a symmetrizable system of partial differential equations of hyperbolic−parabolic nature as will be shown in the following sections.

Remark 2.6. We only investigate in this paper the well-posedness of the Cauchy problem with no vacuum states. On the other hand, various boundary conditions associated with shallow water type equations are discussed by Bresch,[8] Sundbye,[54] Orenga,[45] Li *et al.*,[41] and Sanchez-Hubert and Sanchez-Palencia.[51] Note in particular that, when there are vacuum states at the boundary, no boundary conditions are to be imposed as shown by Sanchez-Hubert and Sanchez-Palencia for vibrating shallow waters[51] using the theory of elliptic degenerate operators of Bouendi and Goulaouic,[6] and by Bresch[8] and Li *et al.*[41] who established in particular that the adherence condition should be written in the form $h\boldsymbol{v} = 0$ so that it disappears when $h = 0$.

3. Quasilinear Form

The governing equations presented in Sec. 2 are recast into a quasilinear vector form in this section.

3.1. *Conservative and natural variables*

The conservative variable U associated with Eqs. (2.1)−(2.4) is given by

$$U = (h, hu, hv, he^{\mathrm{tot}})^t, \tag{3.1}$$

and the natural variable Z by

$$Z = (h, u, v, T)^t, \tag{3.2}$$

where h is the vertical height of the viscous sheet or of the shallow water flow playing the role of a density, u, v are the horizontal components of the mass averaged flow velocity in such a way that the velocity vector is $\boldsymbol{v} = (u, v)^t$, e^{tot} is the total energy per unit mass of the fluid, t is the transposition symbol, and T is the absolute temperature.

The components of U naturally appear as conserved quantities in the Saint-Venant system with an energy equation. On the other hand, the components of the natural variable Z are more practical to use in actual calculations of differential identities.

3.2. *Vector equations*

The Saint-Venant equations modeling thin viscous sheet over fluid substrates or shallow water flows (2.1)–(2.4) can be rewritten in the compact form

$$\partial_t U + \partial_x F_x + \partial_y F_y + \partial_x F_x^{\text{dis}} + \partial_y F_y^{\text{dis}} = \Omega, \tag{3.3}$$

where ∂_t is the time derivative operator, ∂_x, ∂_y are the space derivative operator in the x and y directions respectively, F_x and F_y are the convective fluxes in the x- and y-directions respectively, F_x^{dis} and F_y^{dis} are the dissipative fluxes in the x- and y-directions respectively, and Ω is the source term. We will use the indexing set $C = \{x, y\}$ in the following for the sake of simplicity.

From Sec. 2 the convective fluxes F_x and F_y in the x- and y-directions are given by

$$F_x = (hu, hu^2 + p, huv, he^{\text{tot}}u + pu)^t, \tag{3.4}$$

$$F_y = (hv, hvu, hv^2 + p, he^{\text{tot}}v + pv)^t, \tag{3.5}$$

where p is the pressure and e^{tot} the total energy per unit mass. The dissipative fluxes F_x^{dis} and F_y^{dis} in the x- and y-directions are

$$F_x^{\text{dis}} = (0, \Pi_{xx}, \Pi_{xy}, \mathcal{Q}_x + \Pi_{xx}u + \Pi_{xy}v)^t, \tag{3.6}$$

$$F_y^{\text{dis}} = (0, \Pi_{yx}, \Pi_{yy}, \mathcal{Q}_y + \Pi_{yx}u + \Pi_{yy}v)^t, \tag{3.7}$$

where $\boldsymbol{\Pi}$ is the kinematic viscous stress–tensor (2.9)–(2.10) and $\boldsymbol{Q}$ the kinematic heat flux vector (2.11). Finally, the source term is given by

$$\Omega = (0, 0, 0, \mathcal{H})^t, \tag{3.8}$$

where $\mathcal{H}$ is the heat loss term.

These equations have to be completed by the relations expressing the transport fluxes $\boldsymbol{\Pi}$ and $\boldsymbol{Q}$, the thermodynamic properties p and e^{tot}, and the source term Ω, already presented in Sec. 2. These relations have been given in terms of the natural variable and are used in the following sections to rewrite the system as a quasilinear system in terms of the conservative variable U.

3.3. *Mathematical assumptions*

We describe in this section the mathematical assumptions concerning the thermodynamic properties and the transport coefficients associated with the Saint-Venant equations. These assumptions are assumed to hold in Secs. 3–5.

(Th_1) The fluid density ρ and the pressure factor α are positive constants. The formation energy $e^{\rm st}$ and the formation entropy $s^{\rm st}$ are constants. The specific heat per unit mass c_v, is a C^∞ function of $T \geq 0$ and there exist positive constants $\underline{c}_v$ and $\bar{c}_v$ with $0 < \underline{c}_v \leq c_v(T) \leq \bar{c}_v$, for $T \geq 0$.

(Tr_1) The kinematic shear viscosity ν, the kinematic thermal conductivity $\varkappa$, and the thermal exchange coefficient λ^* are C^∞ functions of T for $T > 0$.

(Tr_2) The kinematic thermal conductivity $\varkappa$, the kinematic shear viscosity ν, and the heat exchange coefficient λ^* are positive functions.

Remark 3.1. The adiabatic situation where $\lambda^* = 0$ is also easily investigated and we only assume that $\lambda^* > 0$ in order to simplify the formal presentation. Similarly, the situations where ν and $\varkappa$ are functions of both T and h are easily taken into account.

3.4. *Dissipation matrices and quasilinear system*

In this section, we rewrite the system of partial differential equations (3.3) as a quasilinear system of second-order partial differential equations in terms of the conservative variable U. In order to express the natural variable Z in terms of the conservative variable U, we first investigate the map $Z \to U$ and its range.

Proposition 3.2. *The map $Z \mapsto U$ is a C^∞ diffeomorphism from the open set $\mathcal{O}_Z = (0,\infty) \times \mathbb{R}^2 \times (0,\infty)$ onto an open set $\mathcal{O}_U$. The open set $\mathcal{O}_U$ is convex and given by*

$$\mathcal{O}_U = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4;\ u_1 > 0,\ u_4 - \phi(u_1, u_2, u_3) > 0\}, \tag{3.9}$$

where $\phi : (0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi(u_1, u_2, u_3) = \frac{1}{2}\,\frac{u_2^2 + u_3^2}{u_1} + e^0 u_1 + \frac{1}{2}\alpha u_1^2,$$

and where e^0 is the formation energy of the fluid at zero temperature.

Proof. From Assumption (Th_1) and the expression of thermodynamic properties, we first deduce that the map $Z \to U$ is C^∞ over the domain $\mathcal{O}_Z$. On the other hand, it is straightforward to show that the map $Z \to U$ is one-to-one and that

$$\partial_Z U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u & h & 0 & 0 \\ v & 0 & h & 0 \\ e^{\rm tot} + \frac{1}{2}\alpha h & hu & hv & hc_v \end{pmatrix},$$

so that the matrix $\partial_Z U$ is nonsingular over $\mathcal{O}_Z$. From the inverse function theorem, we deduce that $Z \to U$ is a C^∞ diffeomorphism onto an open set $\mathcal{O}_U$. From $he^{\rm tot} = he + \frac{1}{2}h\boldsymbol{v}\cdot\boldsymbol{v}$, the expressions of e, and (Th_1), it is then established that $\mathcal{O}_U$ is given by (3.9). The convexity of $\mathcal{O}_U$ is finally a consequence of the convexity of ϕ, which is established by evaluating its second derivative. More specifically, for $u_1 > 0$ and $u_2, u_3 \in \mathbb{R}$, we have

$$\partial^2_{u_1}\phi = \frac{u_2^2 + u_3^2}{u_1^3} + \alpha, \quad \partial^2_{u_1u_2}\phi = -\frac{u_2}{u_1^2}, \quad \partial^2_{u_1u_3}\phi = -\frac{u_3}{u_1^2},$$

$$\partial^2_{u_2}\phi = \partial^2_{u_3}\phi = \frac{1}{u_1}, \quad \partial^2_{u_2u_3}\phi = 0,$$

in such a way that for any $(x_1, x_2, x_3) \in \mathbb{R}^3$ we have

$$\sum_{1\le i,j\le 3} x_i x_j \partial^2_{u_iu_j}\phi = \alpha x_1^2 + \frac{1}{u_1}\left(x_2 - \frac{u_2}{u_1}x_1\right)^2 + \frac{1}{u_1}\left(x_3 - \frac{u_3}{u_1}x_1\right)^2,$$

and the matrix $(\partial^2_{u_iu_j}\phi)_{1\le i,j\le 3}$ is positive definite over $(0,\infty)\times\mathbb{R}^2$. $\square$

In Sec. 2.3, the transport fluxes $\boldsymbol{\Pi}$ and $\boldsymbol{Q}$ and, therefore, the dissipative fluxes $F_x^{\rm dis}$ and $F_y^{\rm dis}$, have been expressed in terms of the gradient of the natural variable Z. By using Proposition 3.2, these dissipation fluxes can thus be expressed as functions of the conservative variable gradients

$$F_i^{\rm dis} = -\sum_{j\in C} B_{ij}(U)\partial_j U,$$

where $C = \{x, y\}$ and $B_{ij}(U)$, $i, j \in C$, are the dissipation matrices. The matrix $B_{ij}(U)$ is a square matrix of dimension 4, which relates the dissipative flux in direction i to the gradient of U in direction j.

We may further introduce the Jacobian matrices A_i, $i \in C$, of the convective fluxes F_i, $i \in C$, defined by

$$A_i = \partial_U F_i, \quad i \in C,$$

and finally rewrite the system (3.3) in the quasilinear form

$$\partial_t U + \sum_{i\in C} A_i(U)\partial_i U = \sum_{i,j\in C} \partial_i(B_{ij}(U)\partial_j U) + \Omega(U), \tag{3.10}$$

where the matrix coefficients are defined on the open convex set $\mathcal{O}_U$. As a direct consequence of (Th_1) and (Tr_1), the system coefficients satisfy the following property (Edp_1)

(Edp_1) The convective fluxes F_i, $i \in C$, the dissipation matrices B_{ij}, $i, j \in C$, and the source term Ω are smooth functions of the variable $U \in \mathcal{O}_U$.

Expanding the sums over $C = \{x, y\}$, these equations can also be written in the more explicit form

$$\begin{aligned}
&\partial_t U + A_x(U)\partial_x U + A_y(U)\partial_y U \\
&\quad = \partial_x(B_{xx}(U)\partial_x U) + \partial_x(B_{xy}(U)\partial_y U) \\
&\qquad + \partial_y(B_{yx}(U)\partial_x U) + \partial_y(B_{yy}(U)\partial_y U) + \Omega(U).
\end{aligned} \tag{3.11}$$

The detailed form of the coefficient matrices $A_i(U)$, $i \in C$, and $B_{ij}(U)$, $i, j \in C$, will not be needed in the following, and, therefore, will not be given.

4. Symmetrization of Saint-Venant Equations

For hyperbolic systems of conservation laws, the existence of a conservative symmetric formulation has been shown to be equivalent to the existence of an entropy function.[17,28,44] These results have been generalized to the case of second-order quasilinear systems of equations by Kawashima and Shizuta.[20,25,38] Kawashima and Shizuta[38] have also shown that, when the nullspace naturally associated with dissipation matrices is a fixed subspace, a symmetric system of conservation equations can be put into a normal form, that is, in the form of a symmetric hyperbolic–parabolic composite system. Giovangigli and Massot[20,25] have further characterized all possible normal forms for such systems.

In this section, we investigate the symmetrization of the Saint-Venant system with an energy equation (3.10). We exhibit a mathematical entropy function and derive the corresponding conservative symmetric form. This symmetric form is then used to derive a normal form. The symmetrizing variable is obtained from the entropy and not from the kinetic energy as investigated by Tadmor,[55] Hauke,[29] and Carey[12] in the isothermal case. These symmetrized systems may also be useful for finite element discretizations and numerical simulations as discussed by Chalot, Hughes and Shakib,[13] Hughes, Franca and Mallet,[33] Hauke[29] and Carey.[12] The assumptions concerning thermodynamic properties (Th_1) and transport properties (Tr_1), (Tr_2) are assumed to hold in this section.

4.1. *Entropy and symmetric conservative form*

The following definition of a symmetric (conservative) form for the system (3.10) is adapted from Kawashima and Shizuta.[20,25,38]

Definition 4.1. Consider a C^∞ dipheomorphism $U \to V$ from the open domain $\mathcal{O}_U$ onto an open domain $\mathcal{O}_V$ and consider the system in the V variable

$$\tilde{A}_0(V)\partial_t V + \sum_{i\in C} \tilde{A}_i(V)\partial_i V = \sum_{i,j\in C} \partial_i(\tilde{B}_{ij}(V)\partial_j V) + \tilde{\Omega}(V), \tag{4.1}$$

where

$$\begin{cases} \tilde{A}_0 = \partial_V U, & \tilde{A}_i = A_i \partial_V U = \partial_V F_i, \\ \tilde{B}_{ij} = B_{ij}\partial_V U, & \tilde{\Omega} = \Omega. \end{cases} \tag{4.2}$$

The system is said of the symmetric form if the matrices $\tilde{A}_0$, $\tilde{A}_i$, $i \in C$, and $\tilde{B}_{ij}$, $i, j \in C$, satisfy the following properties (S_1)–(S_4).

(S_1) The matrix $\tilde{A}_0$ is symmetric positive definite for $V \in \mathcal{O}_V$.
(S_2) The matrices $\tilde{A}_i$, $i \in C$, are symmetric for $V \in \mathcal{O}_V$.
(S_3) We have $\tilde{B}_{ij}^t = \tilde{B}_{ji}$ for $i, j \in C$, and $V \in \mathcal{O}_V$.
(S_4) The matrix $\tilde{B} = \sum_{i,j\in C} \tilde{B}_{ij}(V) w_i w_j$ is symmetric and positive semidefinite, for $V \in \mathcal{O}_V$ and $w \in \Sigma^1$, where Σ^1 is the unit sphere in two dimensions.

The following generalized definition of a mathematical entropy function is adapted[20,25] from Kawashima[37] and Kawashima and Shizuta.[38]

Definition 4.2. Consider a C^∞ function $\sigma(U)$ defined over the open convex domain $\mathcal{O}_U$. The function σ is said to be an entropy function for the system (3.10) if the following properties hold:

(E_1) The function σ is a strictly convex function of $U \in \mathcal{O}_U$ in the sense that the Hessian matrix $\partial_U^2 \sigma$ is positive definite over $\mathcal{O}_U$.
(E_2) There exists real-valued C^∞ functions $q_i = q_i(U)$ such that

$$(\partial_U \sigma) A_i = \partial_U q_i, \quad i \in C, \quad U \in \mathcal{O}_U.$$

(E_3) We have the property that, for any $U \in \mathcal{O}_U$

$$\left(\partial_U^2 \sigma\right)^{-1} B_{ji}^t = B_{ij} \left(\partial_U^2 \sigma\right)^{-1}, \quad i, j \in C.$$

(E_4) The matrix $\tilde{B} = \sum_{i,j\in C} B_{ij}(U) \left(\partial_U^2 \sigma(U)\right)^{-1} w_i w_j$ is symmetric positive semidefinite for any $U \in \mathcal{O}_U$ and any $w \in \Sigma^1$.

Kawashima and Shizuta have established[20,25,38] the equivalence between conservative symmetrizability and the existence of an entropy function.

Theorem 4.3. *The system* (3.10) *admits an entropy function σ defined over the open convex set $\mathcal{O}_U$ if and only if it can be symmetrized over the open convex set $\mathcal{O}_U$. In this situation the symmetrizing variable V and the entropy function can be chosen such that*

$$V = (\partial_U \sigma)^t. \tag{4.3}$$

As is usual for compressible gases,[38] mixtures of reacting gases,[20,25] ambipolar plasmas,[23] we define the mathematical entropy function σ of the Saint-Venant system with an energy equation as the opposite of the physical entropy hs

$$\sigma = -hs,$$

where s is the entropy per unit mass of the fluid under consideration (2.8). The mathematical entropy σ is associated with the physical entropy per unit surface hs and not the entropy per unit volume as usual. The corresponding entropic variable

$$V = (\partial_U \sigma)^t,$$

is then easily evaluated as

$$V = \frac{1}{T}\left(g - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, u, v, -1\right)^t,$$

where g is the Gibbs function.

Proposition 4.4. *The change of variable $U \mapsto V$ is a C^∞ diffeomorphism from the open convex set $\mathcal{O}_U$ onto an open set $\mathcal{O}_V$. The open set $\mathcal{O}_V$ is given by*

$$\mathcal{O}_V = \{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4; u_4 < 0,\ u_1 - \psi(u_2, u_3, u_4) > 0\}, \tag{4.4}$$

where $\psi : \mathbb{R}^2 \times (-\infty, 0) \to \mathbb{R}$ is given by

$$\psi(u_2, u_3, u_4) = -u_4\bar{g}(-1/u_4) + \frac{1}{2}\frac{u_2^2 + u_3^2}{u_4},$$

and where the Gibbs function has been decomposed into $g(h,T) = \bar{g}(T) + \alpha h$.

Proof. From Proposition 3.2, the map $Z \to U$ is a C^∞ diffeomorphism from $\mathcal{O}_Z$ onto $\mathcal{O}_U$, so that we only have to show that the map $Z \to V$ is a C^∞ diffeomorphism from $\mathcal{O}_Z$ onto the open set $\mathcal{O}_V$. From Assumption (Th_1) and the expression of thermodynamic properties, we first deduce that the map $Z \to V$ is C^∞ over the domain $\mathcal{O}_Z$. It is then straightforward to show that the map $Z \to V$ is one-to-one and that its range is $\mathcal{O}_V$ since the Gibbs function can be decomposed in the form $g(h,T) = \bar{g}(T) + \alpha h$. In addition, the matrix $\partial_Z V$ is easily shown to be nonsingular over $\mathcal{O}_Z$ from its triangular structure and the proof is complete, thanks to the inverse function theorem. $\square$

The conservative symmetric form is now investigated in the following theorem.

Theorem 4.5. *The function σ is a mathematical entropy for the system* (3.10), *that is, σ satisfies Properties* (E_1)–(E_4) *of Definition* 4.2. *The symmetrized system associated with the entropic variable $V \in \mathcal{O}_V$ can be written*

$$\tilde{A}_0\partial_t V + \sum_{i\in C}\tilde{A}_i\partial_i V = \sum_{i,j\in C}\partial_i\big(\tilde{B}_{ij}\partial_j V\big) + \tilde{\Omega}, \tag{4.5}$$

and satisfies Properties (S_1)–(S_4) *of Definition* 4.1. *The matrix $\tilde{A}_0$ is given by*

$$\tilde{A}_0 = \frac{T}{\alpha}\begin{pmatrix} 1 & & \mathrm{Sym} \\ \boldsymbol{v} & \boldsymbol{v}\otimes\boldsymbol{v} + \alpha h\boldsymbol{I} & \\ e^{\mathrm{tot}} + \dfrac{1}{2}\alpha h & \left(e^{\mathrm{tot}} + \dfrac{3}{2}\alpha h\right)\boldsymbol{v}^t & \Upsilon_0 \end{pmatrix},$$

where

$$\Upsilon_0 = \left(e^{\mathrm{tot}} + \frac{1}{2}\alpha h\right)^2 + \alpha h(u^2 + v^2 + Tc_v).$$

Since this matrix is symmetric, we only give its block lower triangular part and write "Sym" in the upper triangular part. Denoting by $\boldsymbol{\xi} = (\xi_x, \xi_y)^t$ *an arbitrary vector of* $\mathbb{R}^2$, *the matrices* $\tilde{A}_i$, $i \in C$, *are given by*

$$\sum_{i\in C} \tilde{A}_i \xi_i = \frac{T}{\alpha} \begin{pmatrix} \boldsymbol{v}\cdot\boldsymbol{\xi} & & \mathrm{Sym} \\ \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\xi}) + \alpha h \boldsymbol{\xi} & \Sigma_{\boldsymbol{v},\boldsymbol{v}} & \\ \left(e^{\mathrm{tot}} + \frac{3}{2}\alpha h\right)\boldsymbol{v}\cdot\boldsymbol{\xi} & \Sigma_{e,\boldsymbol{v}} & \Upsilon_1 \boldsymbol{v}\cdot\boldsymbol{\xi} \end{pmatrix},$$

where

$$\Sigma_{\boldsymbol{v},\boldsymbol{v}} = \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}\otimes\boldsymbol{v} + \alpha h(\boldsymbol{v}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes\boldsymbol{v} + 2\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{I}),$$

$$\Sigma_{e,\boldsymbol{v}} = \left(e^{\mathrm{tot}} + \frac{5}{2}\alpha h\right)\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}^t + \left(e^{\mathrm{tot}} + \frac{1}{2}\alpha h\right)\alpha h \boldsymbol{\xi}^t,$$

$$\Upsilon_1 = \left(e^{\mathrm{tot}} + \frac{5}{2}\alpha h\right)\left(e^{\mathrm{tot}} + \frac{1}{2}\alpha h\right) + \alpha h(u^2 + v^2 + Tc_v).$$

The dissipation matrices, are given by

$$\tilde{B}_{xx} = hT\nu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4u \\ 0 & 0 & 1 & v \\ 0 & 4u & v & \theta + 3u^2 \end{pmatrix}, \quad \tilde{B}_{xy} = hT\nu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2v \\ 0 & 1 & 0 & u \\ 0 & v & 2u & 3uv \end{pmatrix},$$

$$\tilde{B}_{yx} = hT\nu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & v \\ 0 & 2 & 0 & 2u \\ 0 & 2v & u & 3uv \end{pmatrix}, \qquad \tilde{B}_{yy} = hT\nu \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & u \\ 0 & 0 & 4 & 4v \\ 0 & u & 4v & \theta + 3v^2 \end{pmatrix},$$

where $\theta = \varkappa T/\nu + (u^2 + v^2)$. *Denoting by* $\boldsymbol{\xi} = (\xi_x, \xi_y)^t$ *and* $\boldsymbol{\zeta} = (\zeta_x, \zeta_y)^t$ *arbitrary vectors of* $\mathbb{R}^2$, *we have*

$$\sum_{i,j\in C} \tilde{B}_{ij}\xi_i\zeta_j = hT\nu \begin{pmatrix} 0 & 0 & \mathrm{Sym} \\ 0 & 2\boldsymbol{\xi}\otimes\boldsymbol{\zeta} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I} & \\ 0 & 2\boldsymbol{v}\cdot\boldsymbol{\zeta}\boldsymbol{\xi}^t + \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{\zeta}^t + \boldsymbol{\zeta}\cdot\boldsymbol{\xi}\boldsymbol{v}^t & \theta\boldsymbol{\zeta}\cdot\boldsymbol{\xi} + 3\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}\cdot\boldsymbol{\zeta} \end{pmatrix}.$$

Finally, the source term $\tilde{\Omega}$ *is given by*

$$\tilde{\Omega} = \Omega.$$

Proof. The calculation of the matrices $\tilde{A}_0$, $\tilde{A}_i$, $i \in C$, and $\tilde{B}_{ij}$, $i, j \in C$, is lengthy but straightforward and, therefore, is omitted. This calculation is easily conducted by using the natural variable Z as an intermediate variable. The symmetry properties of $\tilde{A}_0$, $\tilde{A}_i$, $i \in C$, and $\tilde{B}_{ij}$, $i, j \in C$, required in (S_1)–(S_4) are then obtained.

Consider then a vector $\mathsf{x} \in \mathbb{R}^4$, with components $(\mathsf{x}_h, \mathsf{x}_u, \mathsf{x}_v, \mathsf{x}_T)^t$. After a little algebra, we obtain

$$\begin{aligned}\mathsf{x}^t \tilde{A}_0 \mathsf{x} = \frac{T}{\alpha}\Bigg(&\alpha h(\mathsf{x}_u + u\mathsf{x}_T)^2 + \alpha h(\mathsf{x}_v + v\mathsf{x}_T)^2 \\ &+ \left(\mathsf{x}_h + u\mathsf{x}_u + v\mathsf{x}_v + \left(e^{\mathrm{tot}} + \frac{1}{2}\alpha h\right)\mathsf{x}_T\right)^2 + \alpha h c_v T \mathsf{x}_T^2\Bigg),\end{aligned} \tag{4.6}$$

so that from (Th_1) and the positivity of α, c_v and T, we deduce that $\tilde{A}_0$ is positive definite. Furthermore, a straightforward calculation leads to the following expression

$$\begin{aligned}\mathsf{x}^t \tilde{B}(V, \boldsymbol{w})\mathsf{x} = T\nu h\Big(&3(w_x(\mathsf{x}_u + u\mathsf{x}_T) + w_y(\mathsf{x}_v + v\mathsf{x}_T))^2 \\ &+ (\mathsf{x}_u + u\mathsf{x}_T)^2 + (\mathsf{x}_v + v\mathsf{x}_T)^2 + \frac{\varkappa}{\nu T}\mathsf{x}_T^2\Big),\end{aligned} \tag{4.7}$$

where $\mathsf{x} = (\mathsf{x}_h, \mathsf{x}_u, \mathsf{x}_v, \mathsf{x}_T)^t$ and $w_x^2 + w_y^2 = 1$. The matrix $\tilde{B}$ — easily shown to be symmetric — is thus positive semidefinite from the positivity properties of transport coefficients. Finally, σ also satisfies (E_1)–(E_4) with $q_x = \sigma u$ and $q_y = \sigma v$, as is easily checked and σ is strictly convex since $\tilde{A}_0$ is positive definite over the open convex set $\mathcal{O}_U$. □

We have thus established in this section that the Saint-Venant system satisfies the property:

(Edp_2) The quasilinear Saint-Venant system of partial differential equations (3.10) admits an entropy function σ on the open convex set $\mathcal{O}_U$.

4.2. *Normal forms of Saint-Venant equations*

The quasilinear Saint-Venant system of partial differential equations (3.10) has smooth coefficients and admits an entropy function, that is, satisfies the properties (Edp_1) and (Edp_2). Introducing the symmetrizing variable $V = (\partial_U \sigma)^t$, the corresponding symmetric system (4.5) then satisfies Properties (S_1)–(S_4). However, depending on the range of the dissipation matrices $\tilde{B}$, this system lies between the two limit cases of a hyperbolic system and a strongly parabolic system. In order to split the variables between hyperbolic and parabolic variables, we have to put the system into a normal form, in the form of a symmetric hyperbolic–parabolic composite system.[25,37,38]

To this aim, introducing a new variable W, associated with a diffeomorphism $V \to W$ from $\mathcal{O}_V$ onto $\mathcal{O}_W$, changing of variable $V = V(W)$ in (4.5) and multiplying on the left side by the transpose of the matrix $\partial_W V$, we get a new system in the variable W and have the following definition of a normal form.[38]

Definition 4.6. Consider a system in symmetric form, as in Definition 4.1, and a diffeomorphism $V \to W$ from the open set $\mathcal{O}_V$ onto an open set $\mathcal{O}_W$. The system in the new variable W

$$\overline{A}_0(W)\partial_t W + \sum_{i\in C}\overline{A}_i(W)\partial_i W = \sum_{i,j\in C}\partial_i(\overline{B}(W)\partial_j W) + \overline{\mathcal{T}}(W,\partial_x W) + \overline{\Omega}(W), \quad (4.8)$$

where

$$\begin{cases}\overline{A}_0 = (\partial_W V)^t\tilde{A}_0(\partial_W V), & \overline{B}_{ij} = (\partial_W V)^t\tilde{B}_{ij}(\partial_W V),\\ \overline{A}_i = (\partial_W V)^t\tilde{A}_i(\partial_W V), & \overline{\Omega} = (\partial_W V)^t\tilde{\Omega},\\ \overline{\mathcal{T}} = -\sum_{i,j\in C}\partial_i(\partial_W V)^t\tilde{B}_{ij}(\partial_W V)\partial_j W, & \end{cases} \quad (4.9)$$

satisfies properties (S_1)–(S_4) rewritten in terms of overbar quantities. This system is then said to be of the normal form if there exists a partition of $\{1,\dots,4\}$ into $I=\{1,\dots,n_0\}$ and $II=\{n_0+1,\dots,4\}$, such that the following properties hold.

(Nor_1) The matrices $\overline{A}_0$ and $\overline{B}_{ij}$ have the block structure

$$\overline{A}_0 = \begin{pmatrix}\overline{A}_0^{I,I} & 0\\ 0 & \overline{A}_0^{II,II}\end{pmatrix}, \quad \overline{B}_{ij} = \begin{pmatrix}0 & 0\\ 0 & \overline{B}_{ij}^{II,II}\end{pmatrix}.$$

(Nor_2) The matrix $\overline{B}_{ij}^{II,II}(W,w) = \sum_{i,j\in C}\overline{B}_{ij}^{II,II}(W)w_iw_j$ is positive definite for $W\in\mathcal{O}_W$ and $w\in\Sigma^1$.

(Nor_3) Denoting $\boldsymbol{\partial_x} = (\partial_x,\partial_y)^t$, we have

$$\overline{\mathcal{T}}(W,\boldsymbol{\partial_x}W) = (\overline{\mathcal{T}}_I(W,\boldsymbol{\partial_x}W_{II}), \overline{\mathcal{T}}_{II}(W,\boldsymbol{\partial_x}W))^t,$$

where we have used the vector and matrix block structure induced by the partitioning of $\{1,\dots,4\}$ into $I=\{1,\dots,n_0\}$ and $II=\{n_0+1,\dots,4\}$, so that we have $W=(W_I,W_{II})^t$, for instance.

A sufficient condition for system (4.1) to be recast into a normal form is that the nullspace naturally associated with dissipation matrices is a fixed subspace of $\mathbb{R}^4$. This is Condition N introduced by Kawashima and Shizuta. In the following lemma, we establish that the nullspace invariance property holds for the Saint-Venant system of partial differential equations.

Proposition 4.7. *Let $V\in\mathcal{O}_V$, $\boldsymbol{w}=(w_x,w_y)^t\in\Sigma^1$, and denote*

$$\tilde{B}(V,\boldsymbol{w}) = \sum_{i,j\in C}\tilde{B}_{ij}(V)w_iw_j.$$

The nullspace of the matrix $\tilde{B}$ is one-dimensional and given by

$$N(\tilde{B}) = \mathrm{span}(1,0,0,0)^t,$$

and we have $\tilde{B}_{ij}N(\tilde{B})=0$, for $i,j\in C$.

Proof. According to (4.7) the matrix $\tilde{B}$ is positive semidefinite, so that its nullspace is constituted by the vectors x of $\mathbb{R}^4$ such that $x^t\tilde{B}x=0$. Denoting $x=(x_h,x_u,x_v,x_T)^t$ and using (4.7), the null condition $x^t\tilde{B}x=0$ implies that $x_T=0$

and $x_u = x_v = 0$ and conversely. We have thus obtained that the nullspace of $\tilde{B}(V, \boldsymbol{w})$ is one-dimensional and spanned by $(1,0,0,0)^t$, and it is thus independent of $V \in \mathcal{O}_V$ and $\boldsymbol{w} \in \Sigma^1$. Finally, one easily checks that $\tilde{B}_{ij}(1,0,0,0)^t = 0$, for $i, j \in C$. □

We have thus established the following property

(Edp_3) The nullspace of the matrix $\tilde{B}(V, w) = \sum_{i,j\in C} \tilde{B}_{ij}(V) w_i w_j$ does not depend on V and $w \in \Sigma^1$, $\dim(N(\tilde{B})) = 1$, and we have $\tilde{B}_{ij}(V)N(\tilde{B}) = 0$, $i, j \in C$.

We now investigate normal forms for the system (3.10), or, equivalently, for the system (4.5). Since the nullspace of the matrix $\tilde{B}$ is spanned by the first canonical basis vector, the invertible matrix P of Lemma 3.7 of Giovangigli and Massot[25] can be taken to be the unit tensor in $\mathbb{R}^{4,4}$ so that the auxiliary variables are simply $U' = U$ and $V' = V$. Since $U'_I = U_I = h$ and $V'_{II} = V_{II} = (u, v, -1)/T$, we obtain from the general characterization of normal form the following result.

Theorem 4.8. *Any normal form of the system (4.1) is given by a change of variable in the form*

$$W = \left(\phi_I(h), \phi_{II}\left(\frac{u}{T}, \frac{u}{T}, \frac{-1}{T}\right)\right)^t,$$

where ϕ_I and ϕ_{II} are two diffeomorphisms of $\mathbb{R}$ and $\mathbb{R}^3$ respectively, and we have

$$\overline{\mathcal{T}}(W, \partial_x W) = (0, \overline{\mathcal{T}}_{II}(W, \boldsymbol{\partial_x} W_{II}))^t.$$

We can next use the possibility of mixing the parabolic components — the $V'_{II} = V_{II}$ components — established in Theorem 4.8, in order to simplify the analytic expression of the normal variable and, consequently, of the matrix coefficients appearing in the normal form. More specifically, we select the variable $W = Z$

$$W = (h, u, v, T)^t,$$

easily obtained by combining the $V'_{II} = V_{II}$ components and derive the corresponding normal form of the governing equations.

Theorem 4.9. *The system in the variable $W = (W_I, W_{II})^t$, on the open convex set $\mathcal{O}_W = (0, \infty) \times \mathbb{R}^2 \times (0, \infty)$, with hyperbolic variable*

$$W_I = h,$$

and parabolic variable

$$W_{II} = (u, v, T)^t,$$

can be written in the form

$$\overline{A}_0^{I,I} \partial_t W_I + \sum_{i\in C} \overline{A}_i^{I,I} \partial_i W_I + \sum_{i\in C} \overline{A}_i^{I,II} \partial_i W_{II} = 0, \tag{4.10}$$

$$\overline{A}_0^{II,II}\partial_t W_{II} + \sum_{i\in C}\overline{A}_i^{II,I}\partial_i W_I + \sum_{i\in C}\overline{A}_i^{II,II}\partial_i W_{II} = \sum_{i,j\in C}\partial_i(\overline{B}_{ij}^{II,II}\partial_j W_{II}) + \overline{\mathcal{T}}_{II} + \overline{\Omega}_{II}, \tag{4.11}$$

and is of the normal form. The matrix $\overline{A}_0$ is given by

$$\overline{A}_0 = \frac{1}{T}\begin{pmatrix} \alpha & 0 & 0 \\ 0 & h\boldsymbol{I} & 0 \\ 0 & 0 & \dfrac{hc_v}{T} \end{pmatrix}.$$

Denoting by $\boldsymbol{\xi} = (\xi_x, \xi_y)^t$ an arbitrary vector of $\mathbb{R}^2$, the matrices $\overline{A}_x$ and $\overline{A}_y$ are given by

$$\sum_{i\in C}\overline{A}_i\xi_i = \frac{1}{T}\begin{pmatrix} \alpha\boldsymbol{v}\cdot\boldsymbol{\xi} & & \mathrm{Sym} \\ \alpha h\boldsymbol{\xi} & h\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{I} & \\ 0 & 0 & \dfrac{hc_v}{T}\boldsymbol{v}\cdot\boldsymbol{\xi} \end{pmatrix}. \tag{4.12}$$

Denoting by $\boldsymbol{\xi} = (\xi_x, \xi_y)^t$ and $\boldsymbol{\zeta} = (\zeta_x, \zeta_y)^t$ arbitrary vectors of $\mathbb{R}^2$, the dissipation matrices, $\overline{B}_{ij}$ are such that

$$\sum_{i,j\in C}\overline{B}_{ij}\xi_i\zeta_j = \frac{h}{T}\begin{pmatrix} 0 & & \mathrm{Sym} \\ 0_{2\times 1} & \nu(2\boldsymbol{\xi}\otimes\boldsymbol{\zeta} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I}) & \\ 0 & 0_{1\times 2} & \dfrac{\varkappa}{T}\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I} \end{pmatrix},$$

or equivalently

$$\overline{B}_{xx} = \frac{h}{T}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 4\nu & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 0 & 0 & \dfrac{\varkappa}{T} \end{pmatrix}, \quad \overline{B}_{xy} = \frac{h}{T}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2\nu & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\overline{B}_{yx} = \frac{h}{T}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \nu & 0 \\ 0 & 2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \overline{B}_{yy} = \frac{h}{T}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \nu & 0 & 0 \\ 0 & 0 & 4\nu & 0 \\ 0 & 0 & 0 & \dfrac{\varkappa}{T} \end{pmatrix}.$$

The term $\overline{\mathcal{T}}_{II}$ is easily evaluated as

$$\overline{\mathcal{T}}_{II} = -\frac{1}{T^2}(0, \boldsymbol{\Pi}\cdot\boldsymbol{\partial_x}T, \boldsymbol{\Pi}{:}\boldsymbol{\partial_x}\boldsymbol{v} + \boldsymbol{Q}\cdot\boldsymbol{\partial_x}T)^t,$$

whereas the source term $\overline{\Omega} = (\overline{\Omega}_I, \overline{\Omega}_{II})^t = (\partial_W V)^t\Omega$ is given by

$$\overline{\Omega} = \left(0, 0, 0, -\frac{\lambda^*}{T^2}(T - T^{\mathrm{e}})\right)^t.$$

Proof. The calculations are lengthy but straightforward and make use of Theorem 4.5 and Assumptions (Th_1), (Tr_1) and (Tr_2). □

It is remarkable that the Saint-Venant system of partial differential equations with a temperature equation can be recast into a symmetric hyperbolic–parabolic form already obtained for multicomponent reactive compressible flows,[25] partial equilibrium chemistry models,[27] and ambipolar plasmas.[23] In the next section we investigate global existence around equilibrium states and asymptotic stability of the resulting system of partial differential equations. Different existence results could also be obtained for such symmetrized systems, as for instance local existence results, by using the general theory of Volpert and Hudjaev[58] as investigated for instance for total vibrational nonequilibrium flows[26] and anisotropic magnetized plasmas.[24]

5. Global Existence for the Saint-Venant Equations

In the previous sections, we have established that the quasilinear Saint-Venant system of partial differential equations is symmetrizable and can be written in a normal form and we have already established properties (Edp_1)–(Edp_3). In this section we will first investigate the existence of constant equilibrium states or property (Edp_4). We will next investigate the corresponding linearized normal form and linearized source term. We will indeed establish the local dissipativity properties labeled by (Dis_1)–(Dis_4) that will insure the asymptotic stability of equilibrium states.[20,25] In particular, global existence of solution around equilibrium states as well as decrease estimates will be obtained for the quasilinear Saint-Venant system with a temperature equation. We will use the normal variable $W = Z$ introduced in Theorem 4.9 but other normal variables could be used as well.

5.1. *Local dissipative structure*

We remind that the source term Ω is given by $\Omega = (0, 0, 0, -\lambda^*(T - T^{\mathrm{e}}))^t$, where λ^* is a positive coefficient and $T^{\mathrm{e}} > 0$ a positive temperature.

Proposition 5.1. *Let a height $h^{\mathrm{e}} > 0$ and a velocity $\boldsymbol{v}^{\mathrm{e}} = (u^{\mathrm{e}}, v^{\mathrm{e}})^t \in \mathbb{R}^2$ be given. Then the state U^{e} defined by*

$$U^{\mathrm{e}} = (h^{\mathrm{e}}, h^{\mathrm{e}}u^{\mathrm{e}}, h^{\mathrm{e}}v^{\mathrm{e}}, h^{\mathrm{e}}e^{\mathrm{tot}}(h^{\mathrm{e}}, T^{\mathrm{e}}))^t$$

is an equilibrium state

$$\Omega(U^{\mathrm{e}}) = 0,$$

and for this constant stationary state we also have $Z^{\mathrm{e}} = (h^{\mathrm{e}}, u^{\mathrm{e}}, v^{\mathrm{e}}, T^{\mathrm{e}})^t$.

Selecting arbitrarily $Z^{\mathrm{e}} = (h^{\mathrm{e}}, u^{\mathrm{e}}, v^{\mathrm{e}}, T^{\mathrm{e}})^t$ we have established the following property

(Edp_4) There exists a constant equilibrium state U^{e} such that $\Omega(U^{\mathrm{e}}) = 0$.

We will denote by V^{e} and W^{e} the equilibrium states in the variables V and W, respectively. In order to establish a global existence theorem, we further need to investigate the local dissipative structure of the source term.

Proposition 5.2. *The linearized source term $\tilde{L}(V^{\mathrm{e}}) = -(\partial_V \tilde{\Omega})(V^{\mathrm{e}})$ at the stationary state V^{e} constructed in Proposition 5.1 is given by*

$$\tilde{L}(V^{\mathrm{e}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{*\mathrm{e}} T^{\mathrm{e}2} \end{pmatrix}, \tag{5.1}$$

where $\lambda^{*\mathrm{e}} = \lambda^*(T^{\mathrm{e}})$. This matrix $\tilde{L}(V^{\mathrm{e}})$ is symmetric positive semidefinite and satisfies

$$R(\tilde{L}(V^{\mathrm{e}})) = \mathrm{span}(0,0,0,1)^t,$$

in such a way that we have $\Omega(U(V)) = \tilde{\Omega}(V) \in R(\tilde{L}(V^{\mathrm{e}}))$ for all $V \in \mathcal{O}_V$.

Proof. Evaluating the matrix $\tilde{L}(V^{\mathrm{e}})$ is straightforward, $\tilde{L}(V^{\mathrm{e}})$ is positive semidefinite, and obviously $R(\tilde{L}(V^{\mathrm{e}})) = \mathrm{span}(0,0,0,1)^t$. □

Proposition 5.3. *Let $U^{\mathrm{e}} = U(Z^{\mathrm{e}})$ with $Z^{\mathrm{e}} = (h^{\mathrm{e}}, u^{\mathrm{e}}, v^{\mathrm{e}}, T^{\mathrm{e}})^t$ be a constant equilibrium state in $\mathcal{O}_U$ constructed as in Proposition 5.1. Then there exists a neighborhood $\mathfrak{V}$ of V^{e} and a positive constant δ such that*

$$\delta |\tilde{\Omega}(V)|^2 \leq -\langle V - V^{\mathrm{e}}, \tilde{\Omega}(V) \rangle, \quad V \in \mathfrak{V}. \tag{5.2}$$

Proof. From the expression of V, we obtain

$$\langle V - V^{\mathrm{e}}, \tilde{\Omega}(V) \rangle = -\frac{\lambda^*}{T T^{\mathrm{e}}} (T - T^{\mathrm{e}})^2$$

and (5.2) since $|\tilde{\Omega}(V)|^2 = \lambda^{*2}(T - T^{\mathrm{e}})^2$ and λ^* is a positive function. □

We have thus established Properties (Dis_3) and (Dis_4)

(Dis_3) The smallest linear subspace containing the source term vector $\tilde{\Omega}(V)$, for all $V \in \mathcal{O}_V$, is included in the range of $\tilde{L}(V^{*\mathrm{e}})$, with $\tilde{L} = (\partial_V W)^t \overline{L}(V^{*\mathrm{e}}) \partial_V W$.

(Dis_4) There exists a neighborhood of $V^{*\mathrm{e}}$, in $\mathcal{O}_V$, and a positive constant $\delta > 0$ such that, for any V in this neighborhood, we have

$$\delta |\tilde{\Omega}(V)|^2 \leq -\langle V - V^{*\mathrm{e}}, \tilde{\Omega}(V) \rangle.$$

5.2. *Linearized normal form*

If we linearize the symmetric hyperbolic–parabolic system (4.10)–(4.11) around a constant stationary state $W^{\mathrm{e}} = (h^{\mathrm{e}}, u^{\mathrm{e}}, v^{\mathrm{e}}, T^{\mathrm{e}})^t$, we obtain the linear symmetric system

$$\overline{A}_0(W^{\mathrm{e}})\partial_t z + \sum_{i\in C} \overline{A}_i(W^{\mathrm{e}})\partial_i z = \sum_{i,j\in C} \overline{B}_{ij}(W^{\mathrm{e}})\partial_i\partial_j z - \overline{L}(W^{\mathrm{e}})z, \tag{5.3}$$

where the zeroth-order term is defined as $\overline{L}(W^{\mathrm{e}}) = -(\partial_W \overline{\Omega})(W^{\mathrm{e}})$ and is given by

$$\overline{L}(W^{\mathrm{e}}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{*\mathrm{e}} \end{pmatrix}. \tag{5.4}$$

Taking into account that (4.10)–(4.11) is a normal form, and since the matrix $\overline{L}(W^{\mathrm{e}})$ is symmetric positive semidefinite, we obtain that Property (Dis_1) is satisfied

(Dis_1) The matrix $\overline{A}_0(W^{\mathrm{e}})$ is symmetric positive definite, the matrices $\overline{A}_i(W^{\mathrm{e}})$, $i \in C$, are symmetric, we have the reciprocity relations $(\overline{B}_{ij}(W^{\mathrm{e}}))^t = \overline{B}_{ji}(W^{\mathrm{e}})$, $i, j \in C$, and the matrix $\overline{L}(W^{\mathrm{e}})$ is symmetric positive semidefinite.

We next have to investigate the existence of compensating matrices K^j, $j \in C$ as introduced by Kawashima.[25,37,53] In the following proposition, we denote by $\overline{B}(W^{\mathrm{e}}, \boldsymbol{\xi})$ the matrix $\overline{B}(W^{\mathrm{e}}, \boldsymbol{\xi}) = \sum_{i,j\in C} \overline{B}_{ij}\xi_i\xi_j$.

Proposition 5.4. *For a sufficiently small and positive a, the matrices K^j, $j \in C$, defined by*

$$\sum_{j\in C} \xi_j K^j = a \begin{pmatrix} 0 & \boldsymbol{\xi}^t & 0 \\ -\boldsymbol{\xi} & 0_{2\times 2} & 0_{2\times 1} \\ 0 & 0_{1\times 2} & 0 \end{pmatrix} \overline{A}_0(W^{\mathrm{e}})^{-1},$$

where $\boldsymbol{\xi} = (\xi_x, \xi_y)^t$, are compensating matrices. In particular, the products $K^j \overline{A}_0(W^{\mathrm{e}})$ are skew-symmetric and the matrix

$$\sum_{i,j\in C} K^j \overline{A}_i(W^{\mathrm{e}})\xi_i\xi_j + \overline{B}(W^{\mathrm{e}}, \boldsymbol{\xi}),$$

is positive definite for $\boldsymbol{\xi} \in \Sigma^1$.

Proof. It is obvious by construction that the products $K^j \overline{A}_0(W^{\mathrm{e}})$, $j \in C$, are skew-symmetric. On the other hand, a direct calculation yields

$$\sum_{i,j\in C} \xi_j K^j \overline{A}_i(W^{\mathrm{e}})\xi_i = a \begin{pmatrix} \alpha|\boldsymbol{\xi}|^2 & (\boldsymbol{v}^{\mathrm{e}}\cdot\boldsymbol{\xi})\boldsymbol{\xi}^t & 0 \\ -(\boldsymbol{v}^{\mathrm{e}}\cdot\boldsymbol{\xi})\boldsymbol{\xi} & -h^{\mathrm{e}}\boldsymbol{\xi}\otimes\boldsymbol{\xi} & 0_{2\times 1} \\ 0 & 0_{1\times 2} & 0 \end{pmatrix}, \tag{5.5}$$

where the superscript e indicates that the corresponding quantity is evaluated at W^{e}. As a consequence, for $\boldsymbol{\xi} \in \Sigma^1$, and $x = (x_h, x_u, x_v, x_T)^t$, we have $|\boldsymbol{\xi}| = 1$, and there exists $\beta > 0$ such that

$$\left\langle x^t, \sum_{i,j\in C} \xi_j, K^j \overline{A}_i(, W^{\mathrm{e}})\xi_i x \right\rangle \geq \frac{a\alpha}{2}(x_h^2 - \beta(x_u^2 + x_v^2 + x_T^2)).$$

Using now Property (Nor_2), the matrix

$$\sum_{i,j\in C} K^j \overline{A}_i(W^{\mathrm{e}})\xi_i\xi_j + \overline{B}(W^{\mathrm{e}}, \boldsymbol{\xi})$$

is positive definite for $\boldsymbol{\xi} \in \Sigma^1$ and a sufficiently small. □

We have thus established (Dis_2)

(Dis_2) The linearized system is strictly dissipative in the sense that there exists compensating matrices K^j, $j \in C$.

Remark 5.5. Different formulations can be used in order to establish the strict dissipativity of the linearized normal form as investigated by Shizuta and Kawashima.[53] However, we have chosen to directly establish the stronger Proposition 5.4 which implies the existence of a combined compensating matrix $K = \sum_{j\in C} K^j\xi_j$ as discussed by Shizuta and Kawashima.[53]

5.3. *Global existence and asymptotic stability*

In the previous sections, we have established that Properties (Edp_1)$-$(Edp_4) and (Dis_1)$-$(Dis_4) are satisfied. Therefore the existence theorems established in Refs. 20 and 25 can be applied to the system (4.10)$-$(4.11) governing shallow water flows or thin viscous sheets over fluid substrates written in the $W = (W_I, W_{II})^t$ variable, with the hyperbolic variable

$$W_I = h,$$

and the parabolic variable

$$W_{II} = (u, v, T)^t.$$

Theorem 5.6. *Consider the system* (4.10)$-$(4.11) *with* $d = 2$, $l \geq [d/2] + 2$, *and let* $W^0(x)$ *be such that*

$$W^0 - W^{\mathrm{e}} \in W_2^l(\mathbb{R}^d).$$

Then, if $\|W^0 - W^{\mathrm{e}}\|_{l,2}$ *is small enough, there exists a unique global solution to the the Cauchy problem* (4.10)$-$(4.11)

$$\overline{A}_0\partial_t W + \sum_{i\in C} \overline{A}_i\partial_i W = \sum_{i,j\in C} \partial_i(\overline{B}_{ij}\partial_j W) + \overline{\mathcal{T}} + \overline{\Omega},$$

with initial condition

$$W(0,x) = W^0(x),$$

such that

$$\begin{aligned} &W_I - W_I^{\mathrm{e}} \in C^0([0,\infty); W_2^l(\mathbb{R}^d)) \cap C^1([0,\infty); W_2^{l-1}(\mathbb{R}^d)),\\ &W_{II} - W_{II}^{\mathrm{e}} \in C^0([0,\infty); W_2^l(\mathbb{R}^d)) \cap C^1([0,\infty); W_2^{l-2}(\mathbb{R}^d)), \end{aligned} \tag{5.6}$$

and

$$\begin{cases} \boldsymbol{\partial}_{\boldsymbol{x}} W_I \in L^2(0,\infty; W_2^{l-1}(\mathbb{R}^d)),\\ \boldsymbol{\partial}_{\boldsymbol{x}} W_{II} \in L^2(0,\infty; W_2^{l}(\mathbb{R}^d)). \end{cases}$$

Furthermore, W *satisfies the estimate*

$$\begin{aligned} \|W(t) - W^{\mathrm{e}}\|_{l,2}^2 &+ \int_0^t (\|\boldsymbol{\partial}_{\boldsymbol{x}} h(\tau)\|_{l-1,2}^2 + \|\boldsymbol{\partial}_{\boldsymbol{x}} u(\tau)\|_{l,2}^2 \\ &+ \|\boldsymbol{\partial}_{\boldsymbol{x}} v(\tau)\|_{l,2}^2 + \|\boldsymbol{\partial}_{\boldsymbol{x}} T(\tau)\|_{l,2}^2)\, d\tau \le \beta \|W^0 - W^{\mathrm{e}}\|_{l,2}^2, \end{aligned} \tag{5.7}$$

where β *is a positive constant and* $\sup_{x\in\mathbb{R}^d} |W(t) - W^{\mathrm{e}}|$ *goes to zero as* $t \to \infty$.

Theorem 5.7. *Keeping the assumptions of the preceding theorem, assume that* $d = 2$, $l \ge [d/2] + 3$ *and* $W^0 - W^{\mathrm{e}} \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ *with* $p \in [1, 2)$. *Then, if* $\|W(t) - W^{\mathrm{e}}\|_{l-2,2} + \|W(t) - W^{\mathrm{e}}\|_{0,p}$ *is small enough, the unique global solution to the Cauchy problem satisfies for* $t \in [0,\infty)$ *the decay estimate*

$$\|W(t) - W^{\mathrm{e}}\|_{l-2,2} \le \beta(1+t)^{-\gamma}(\|W(t) - W^{\mathrm{e}}\|_{l-2,2} + \|W(t) - W^{\mathrm{e}}\|_{0,p}),$$

where β *is a positive constant and* $\gamma = d \times (1/2p - 1/4)$.

Remark 5.8. Theorems 5.6 and 5.7 are easily adapted to the situation $d = 1$, further assuming that $p = 1$, or to the situation $\lambda^{*\mathrm{e}} = 0$ where T^{e} can then be chosen arbitrarily. Various extra effects like friction forces or wind effects can also easily be taken into account in Theorem 5.6.

Remark 5.9. We have investigated in this paper the Cauchy problem for the equations governing thin viscous sheets over fluid substrates. Similar methods could be applied to investigate strong solutions to Initial-Boundary value problems in the absence of vacuum by imposing classical Navier−Stokes type boundary conditions as discussed for instance by Sundbye,[54] Orenga,[45] Lions,[42] and Bresch.[8] On the other hand, global weak solutions have been investigated by Bresch,[8] Bresch and Desjardins,[9] and Li *et al.*[41] by using gradient entropies, and higher order entropies have also been discussed by Giovangigli.[21;22]

6. A Thin Viscous Sheet Model

We investigate in Secs. 6 and 7 a three-dimensional model of a thin viscous sheet over a fluid substrate and its two-dimensional asymptotic limit.

We first present in this section the three-dimensional partial differential equations governing a thin viscous layer of an incompressible fluid with two free boundaries, an upper fluid/gas boundary and a lower fluid/substrate boundary. The upper gas may depend on a particular application under concern and will be denoted by "gas" for the sake of notational simplicity. On the other hand, the fluid substrate will be modeled by using the hydrostatic approximation. In Sec. 7, we will perform an asymptotic analysis and derive the Saint-Venant equations with an energy equation and temperature-dependent transport coefficients from the three-dimensional governing equations presented in this section.

There are various examples of such viscous layers over fluid substrates as for instance oil slicks over water,[30] float glasses,[47] and Atlantic waters over the deeper denser Mediterranean sea in the Strait of Gilbraltar.[43]

During the spreading of an oil spill, there indeed exist several regimes where it can be modeled as a thin viscous sheet over a water substrate.[46] This is notably the case during the gravity/viscous or viscous/surface-tension spreading regimes.[30] The incompressible oil flow then presents two free boundaries, the upper oil/air interface and the lower oil/sea interface. More refined models may also include other effects like wind dispersion, water currents, shore deposition, evaporation, or dissolution, in order to describe more realistically oil slick trajectories.[46]

In a float glass, molten glass is flowing and floating above molten tin, and is progressively cooled in order to produce plate glass.[31,32,47] This procedure gives the glass sheet a smooth interface and modern windows are made from float glasses. The incompressible molten glass flow then presents two free boundaries, the upper glass/gas interface and the lower glass/tin interface. The reducing atmosphere above the molten glass and the tin bath is typically a mixture of nitrogen and hydrogen to prevent the oxidation of tin.

On the other hand, in the Strait of Gilbraltar, the denser Mediterranean sea flows below Atlantic waters penetrating in the Alboran sea. These phenomena may be modeled by using bi-layer Saint-Venant shallow water equations.[43] More recently, multi-layer Saint-Venant equations have also been investigated.[1,2,36,43]

Nevertheless, we will not discuss a particular application in the following sections since the models investigated may be applied to quite different situations. We will thus generically denote by "fluid" the liquid constitutive of the viscous sheet, by "gas" the gas above the sheet, and by "substrate" or "fluid substrate" the liquid substrate below the sheet.

6.1. *Setting of the problem*

We consider a three-dimensional flow governed by the incompressible Navier–Stokes equations with temperature-dependent transport coefficients. The flow configuration is depicted in Fig. 1 with an upper fluid/gas free boundary and a lower fluid/substrate free boundary. The incompressible fluid constitutive of the viscous sheet is termed the "fluid", the gas above the viscous sheet is termed the "gas", and the

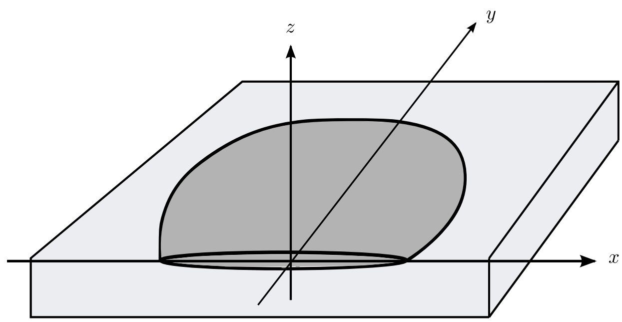

Fig. 1. Schematic of the thin viscous sheet.

lower fluid constitutive of the substrate is termed the "substrate" or the "fluid substrate".

The equations governing the viscous incompressible fluid can be written in the nonconservative form

$$\boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v} = 0,$$

$$\rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v} + \boldsymbol{\partial}_{\mathbf{x}} \mathrm{p} - \boldsymbol{\partial}_{\mathbf{x}} \cdot (\eta \mathbf{d}) = \rho \mathbf{g},$$

$$\rho c_v \partial_t T + \rho c_v \mathbf{v} \cdot \boldsymbol{\partial}_{\mathbf{x}} T - \boldsymbol{\partial}_{\mathbf{x}} \cdot (\lambda T) = \frac{1}{2} \eta \mathbf{d} : \mathbf{d},$$

where $\boldsymbol{\partial}_{\mathbf{x}} = (\partial_x, \partial_y, \partial_z)^t$ is the three-dimensional gradient vector, ρ the constant density of the incompressible fluid, $\mathbf{v} = (u, v, w)^t$ the three-dimensional velocity vector, p the pressure of the three-dimensional glass flow, $\mathbf{g} = (0, 0, \mathrm{g})^t$ the gravity assumed to be constant and vertical, $\mathbf{d} = \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v} + \boldsymbol{\partial}_{\mathbf{x}} \mathbf{v}^t$ the strain tensor, c_v the heat capacity per unit mass of the incompressible fluid, η the fluid viscosity and λ the fluid thermal conductivity of the fluid. We denote by $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ the canonical basis vectors associated with the three-dimensional Cartesian coordinates $\mathbf{x} = (x, y, z)^t$. We will denote by $\boldsymbol{\Pi} = -\eta \mathbf{d}$ the viscous tensor, $\mathbf{P}$ the pressure tensor $\mathbf{P} = \mathrm{p}\mathbf{I} + \boldsymbol{\Pi}$, $\boldsymbol{\sigma} = -\mathbf{P}$ the Cauchy stress tensor, and $\mathbf{Q}$ the heat flux $\mathbf{Q} = -\lambda \boldsymbol{\partial}_{\mathbf{x}} T$. Note that we use italic fonts in order to denote the asymptotic two-dimensional Saint-Venant model and roman fonts in order to denote the original three-dimensional incompressible Navier−Stokes model. We will assume in the following that the pressure in the fluid is measured relative to the atmospheric pressure $\mathrm{p}_{\mathrm{atm}}$ for the sake of simplicity.

The boundary conditions are that of free boundaries at the upper fluid/gas interface $z = h_{\mathrm{gas}}$ and at the lower fluid/substrate interface $z = h_{\mathrm{sub}}$. On the top boundary, the fluid particles stay on the surface $z - h_{\mathrm{gas}} = 0$ and the usual kinematic condition yields that

$$w = \partial_t h_{\mathrm{gas}} + u \partial_x h_{\mathrm{gas}} + v \partial_y h_{\mathrm{gas}},$$

where we have denoted by $\mathbf{v} = (u, v, w)^t$ the three components of the velocity vector $\mathbf{v}$. On the other hand, the dynamic condition at the top free boundary can be written

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{\mathrm{gas}} = \boldsymbol{\sigma}_{\mathrm{gas}} \cdot \mathbf{n}_{\mathrm{gas}} - \gamma_{\mathrm{f/g}} \mathcal{C}_{\mathrm{gas}} \mathbf{n}_{\mathrm{gas}},$$

where $\gamma_{\mathrm{f/g}}$ is the surface tension between the fluid and gas and $\mathcal{C}_{\mathrm{gas}}$ the total curvature of the surface $z = h_{\mathrm{gas}}$ seen from the fluid and given by

$$\mathcal{C}_{\mathrm{gas}} = \frac{\partial_x^2 h_{\mathrm{gas}}(1 + (\partial_y h_{\mathrm{gas}})^2) + \partial_y^2 h_{\mathrm{gas}}(1 + (\partial_x h_{\mathrm{gas}})^2) - 2\partial_x h_{\mathrm{gas}} \partial_y h_{\mathrm{gas}} \partial_{xy}^2 h_{\mathrm{gas}}}{(1 + (\partial_x h_{\mathrm{gas}})^2 + (\partial_y h_{\mathrm{gas}})^2)^{3/2}}.$$

In this dynamic boundary condition, the outward normal vector at the fluid/gas interface is given by

$$\mathbf{n}_{\mathrm{gas}} = ((\partial_x h_{\mathrm{gas}})^2 + (\partial_y h_{\mathrm{gas}})^2 + 1)^{-1/2}(-\partial_x h_{\mathrm{gas}}, -\partial_y h_{\mathrm{gas}}, 1)^t,$$

and the stress−tensor in gas by

$$\boldsymbol{\sigma}_{\mathrm{gas}} \cdot \mathbf{n}_{\mathrm{gas}} = -\mathrm{p}_{\mathrm{atm}} \mathbf{n}_{\mathrm{gas}},$$

where $\mathrm{p}_{\mathrm{atm}}$ denotes the atmospheric pressure.

Remark 6.1. A more general dynamic boundary condition taking into account the spatial variations of the surface tension $\gamma_{\mathrm{f/g}}$ can be written in the form

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{\mathrm{gas}} = \boldsymbol{\sigma}_{\mathrm{gas}} \cdot \mathbf{n}_{\mathrm{gas}} - \gamma_{\mathrm{f/g}} \mathcal{C}_{\mathrm{gas}} \mathbf{n}_{\mathrm{gas}} - (\mathbf{I} - \mathbf{n}_{\mathrm{gas}} \otimes \mathbf{n}_{\mathrm{gas}}) \boldsymbol{\partial}_{\mathbf{x}} \gamma_{\mathrm{f/g}},$$

where $\boldsymbol{\partial}_{\mathbf{x}} \gamma_{\mathrm{f/g}}$ denotes the gradient of the surface tension $\gamma_{\mathrm{f/g}}$. However, for the sake of simplicity, we will assume in the following that $\gamma_{\mathrm{f/g}}$ is a constant.

Similarly, at the lower boundary, $z = h_{\mathrm{sub}}$, the vertical velocity component w is given by

$$w = \partial_t h_{\mathrm{sub}} + u \partial_x h_{\mathrm{sub}} + v \partial_y h_{\mathrm{sub}},$$

and the dynamic condition reads

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{\mathrm{sub}} = \boldsymbol{\sigma}_{\mathrm{sub}} \cdot \mathbf{n}_{\mathrm{sub}} - \gamma_{\mathrm{f/s}} \mathcal{C}_{\mathrm{sub}} \mathbf{n}_{\mathrm{sub}},$$

where $\gamma_{\mathrm{f/s}}$ is the surface tension between the fluid and the substrate, $\mathcal{C}_{\mathrm{sub}}$ the total curvature of the surface $z = h_{\mathrm{sub}}$ seen from the fluid, and

$$\mathbf{n}_{\mathrm{sub}} = ((\partial_x h_{\mathrm{sub}})^2 + (\partial_y h_{\mathrm{sub}})^2 + 1)^{-1/2}(\partial_x h_{\mathrm{sub}}, \partial_y h_{\mathrm{sub}}, -1)^t,$$

the outward unit normal vector at the fluid/substrate interface. Thanks to the hydrostatic approximation, the normal component of the stress−tensor in the fluid substrate is given by

$$\boldsymbol{\sigma}_{\mathrm{sub}} \cdot \mathbf{n}_{\mathrm{sub}} = -\mathrm{p}_{\mathrm{sub}} \mathbf{n}_{\mathrm{sub}},$$

where $\mathrm{p}_{\mathrm{sub}}$ denotes the pressure in the substrate flow given by

$$\mathrm{p}_{\mathrm{sub}} = \mathrm{p}_{\mathrm{atm}} + \rho_{\mathrm{sub}} \mathrm{g} h_{\mathrm{sub}}.$$

From a thermal point of view, at the top and bottom interfaces, we have the boundary conditions

$$-\lambda \boldsymbol{\partial}_{\mathbf{x}} T \cdot \mathbf{n}_{\mathrm{gas}} = \lambda_{\mathrm{gas}}^*(T - T_{\mathrm{gas}}),$$

$$-\lambda \boldsymbol{\partial}_{\mathbf{x}} T \cdot \mathbf{n}_{\mathrm{sub}} = \lambda_{\mathrm{sub}}^*(T - T_{\mathrm{sub}}),$$

where T_{gas} and T_{sub} are given temperatures in the gas and in the substrate flows, respectively, and where λ^*_{gas} and λ^*_{sub} are the heat exchange coefficients.

6.2. *Rescaled equations*

In order to perform an asymptotic analysis of the three-dimensional incompressible fluid flow, we need to specify the order of magnitude of the various terms appearing in the governing equations. For this purpose, for each quantity ϕ, we introduce a typical order of magnitude denoted by $\langle\phi\rangle$. We introduce in particular a characteristic horizontal length $\langle x\rangle = \langle y\rangle$ and vertical length $\langle z\rangle = \epsilon\langle x\rangle$ where the aspect ratio ϵ is the small parameter associated with the thickness of the fluid viscous sheet. We correspondingly introduce a characteristic horizontal velocity $\langle u\rangle = \langle v\rangle$ and vertical velocity $\langle w\rangle = \epsilon\langle u\rangle$ as well as a characteristic density $\langle\rho\rangle = \rho$ where ρ is the constant density of the fluid constitutive of the viscous sheet. Denoting by $\langle\eta\rangle$ a characteristic viscosity, the Reynolds number Re is then given by

$$\mathrm{Re} = \frac{\langle\rho\rangle\langle u\rangle\langle x\rangle}{\langle\eta\rangle}. \tag{6.1}$$

We define the characteristic time from the characteristic length $\langle x\rangle$ and the characteristic velocity $\langle u\rangle$ by letting $\langle t\rangle = \langle x\rangle/\langle u\rangle$. Denoting by $\langle c_v\rangle$ a typical heat capacity and $\langle\lambda\rangle$ a characteristic heat conductivity of the fluid, the characteristic internal energy is defined by $\langle e\rangle = \langle c_v\rangle\langle T\rangle$ and the Prandtl number Pr by

$$\mathrm{Pr} = \frac{\langle\eta\rangle\langle c_v\rangle}{\langle\lambda\rangle}. \tag{6.2}$$

Note that $c_p = c_v$ for an incompressible fluid and that we may set for instance $\langle c_v\rangle = R/m$, where R is the perfect gas constant and m the molar mass of the incompressible fluid. We will also denote by Ec the energy ratio or Eckert number

$$\mathrm{Ec} = \frac{\langle u\rangle^2}{\langle c_v\rangle\langle T\rangle}. \tag{6.3}$$

For a fluid, this number plays a similar role as that of the square of the Mach number for a gas. From these definitions we obtain that $\langle\eta\rangle = \langle\rho\rangle\langle u\rangle\langle x\rangle/\mathrm{Re}$, $\langle e\rangle = \langle c_v\rangle\langle T\rangle = \langle u\rangle^2/\mathrm{Ec}$ and $\langle\lambda\rangle = \langle\rho\rangle\langle u\rangle\langle x\rangle\langle c_v\rangle/(\mathrm{RePr})$. We define the characteristic pressure as $\langle \mathrm{p}\rangle = \langle\rho\rangle\langle u\rangle^2$ and the Froude number by

$$\mathrm{Fr} = \frac{\langle u\rangle^2}{\langle \mathrm{g}\rangle\langle x\rangle}, \tag{6.4}$$

so that $\langle \mathrm{g}\rangle = \langle u\rangle^2/(\mathrm{Fr}\langle x\rangle)$. Denoting by $\langle\gamma\rangle$ a typical surface tension, the capillary number is defined by

$$\mathrm{Ca} = \frac{\langle\eta\rangle\langle u\rangle}{\langle\gamma\rangle}, \tag{6.5}$$

so that $\langle\gamma\rangle = \langle x\rangle\langle\rho\rangle\langle u\rangle^2/\mathrm{ReCa}$. We also introduce a typical heat exchange coefficient $\langle\lambda^*\rangle$ and the reduced quantity

$$\mathrm{Ex} = \frac{\langle\lambda^*\rangle\langle x\rangle}{\langle\lambda\rangle}. \tag{6.6}$$

In the asymptotic analysis, performed in the next section, it will be assumed that

$$\mathrm{Fr} = \epsilon\overline{\mathrm{Fr}}, \quad \mathrm{Ca} = \frac{\overline{\mathrm{Ca}}}{\epsilon}, \quad \mathrm{Ex} = \epsilon\overline{\mathrm{Ex}}, \tag{6.7}$$

and that the numbers Re, Pr, Ec, $\overline{\mathrm{Fr}}$, $\overline{\mathrm{Ca}}$ and $\overline{\mathrm{Ex}}$ are of zeroth-order with respect to ϵ, that is, are finite as $\epsilon \to 0$. Assuming that $\overline{\mathrm{Ex}}$ and $1/\overline{\mathrm{Ca}}$ are small means that surface tension effects as well as thermal exchanges are corrective effects. In order to simplify the formal presentation, it will be convenient to define the modified reduced quantities

$$\overline{\eta} = \frac{\widehat{\eta}}{\mathrm{Re}}, \quad \overline{\lambda} = \frac{\widehat{\lambda}}{\mathrm{PrRe}}, \tag{6.8}$$

$$\overline{\mathrm{g}} = \frac{\widehat{\mathrm{g}}}{\overline{\mathrm{Fr}}}, \quad \overline{\gamma} = \frac{\widehat{\gamma}}{\mathrm{Re}\,\overline{\mathrm{Ca}}}, \quad \overline{\lambda}^* = \frac{\widehat{\lambda}^*\,\overline{\mathrm{Ex}}}{\mathrm{PrRe}}. \tag{6.9}$$

These quantities are such that

$$\eta = \langle\rho\rangle\langle u\rangle\langle x\rangle\,\overline{\eta}, \quad \lambda = \langle\rho\rangle\langle u\rangle\langle x\rangle\langle c_v\rangle\,\overline{\lambda}, \tag{6.10}$$

$$\gamma = \epsilon\langle\rho\rangle\langle u\rangle^2\langle x\rangle\,\overline{\gamma}, \quad g = \langle u\rangle^2\,\overline{\mathrm{g}}/\langle x\rangle, \tag{6.11}$$

$$\lambda^* = \epsilon\langle\rho\rangle\langle u\rangle\langle c_v\rangle\,\overline{\lambda}^*, \tag{6.12}$$

and will simplify the formal presentation of the asymptotic analysis. From the aspect ratio of the thin viscous sheet, we also deduce that the curvature is typically of the order $\langle\mathcal{C}\rangle = \epsilon/\langle x\rangle$. We will also denote by a the density ratio

$$\mathsf{a} = \frac{\rho}{\rho_{\mathrm{sub}}}, \tag{6.13}$$

where ρ is the density of the incompressible fluid constitutive of the viscous sheet and ρ_{sub} the density of the incompressible substrate fluid.

Remark 6.2. Typical values for density ratios are $\mathsf{a} \simeq 0.70{-}0.97$ between crude oil and water for oil slicks, $\mathsf{a} \simeq 0.35$ between glass and tin for float glasses, $\mathsf{a} \simeq 0.997$ between Atlantic and Mediterranean waters. Typical aspect ratios are $\epsilon \simeq 10^{-9}{-}10^{-6}$ for oil slicks, and $\epsilon \simeq 10^{-3}$ for float glasses.

Upon defining the reduced quantity $\widehat{\phi} = \phi/\langle\phi\rangle$ associated with each quantity ϕ of the fluid model, we can now estimate the order of magnitude of each term in the governing partial differential equations. Using the general notation for rescaled variables, the reduced equations can be written in the form

$$\boldsymbol{\partial}_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}} = 0, \tag{6.14}$$

$$\partial_{\widehat{t}} \widehat{\mathbf{v}} + \widehat{\mathbf{v}} \cdot \boldsymbol{\partial}_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}} + \boldsymbol{\partial}_{\widehat{\mathbf{x}}} \widehat{\mathrm{p}} - \boldsymbol{\partial}_{\widehat{\mathbf{x}}} \cdot (\overline{\eta} \widehat{\mathbf{d}}) = \overline{\mathbf{g}}, \tag{6.15}$$

$$\widehat{c}_v \partial_{\widehat{t}} \widehat{T} + \widehat{c}_v \widehat{\mathbf{v}} \cdot \boldsymbol{\partial}_{\widehat{\mathbf{x}}} \widehat{T} - \boldsymbol{\partial}_{\widehat{\mathbf{x}}} \cdot (\overline{\lambda} \widehat{T}) = \overline{\Phi}, \tag{6.16}$$

where $\overline{\Phi} = \frac{1}{2} \overline{\eta} \widehat{\mathbf{d}} : \widehat{\mathbf{d}} / \mathrm{Ec}$ is the reduced viscous dissipation term. In order to perform an asymptotic expansion of all the flow variables, it is further necessary to explicit the governing equations in the horizontal and vertical directions. Upon expanding the flow vector equations and dividing the vertical momentum equation by the aspect ratio ϵ we obtain that

$$\partial_{\widehat{x}} \widehat{u} + \partial_{\widehat{y}} \widehat{v} + \partial_{\widehat{z}} \widehat{w} = 0, \tag{6.17}$$

$$\begin{aligned} &\partial_{\widehat{t}} \widehat{u} + \widehat{u} \partial_{\widehat{x}} \widehat{u} + \widehat{v} \partial_{\widehat{y}} \widehat{u} + \widehat{w} \partial_{\widehat{z}} \widehat{u} - \partial_{\widehat{x}} (2 \overline{\eta} \partial_{\widehat{x}} \widehat{u}) \\ &\quad - \partial_{\widehat{y}} (\overline{\eta} (\partial_{\widehat{y}} \widehat{u} + \partial_{\widehat{x}} \widehat{v})) - \frac{1}{\epsilon^2} \partial_{\widehat{z}} (\overline{\eta} \partial_{\widehat{z}} \widehat{u}) - \partial_{\widehat{z}} (\overline{\eta} \partial_{\widehat{x}} \widehat{w}) + \partial_{\widehat{x}} \widehat{\mathrm{p}} = 0, \end{aligned} \tag{6.18}$$

$$\begin{aligned} &\partial_{\widehat{t}} \widehat{v} + \widehat{u} \partial_{\widehat{x}} \widehat{v} + \widehat{v} \partial_{\widehat{y}} \widehat{v} + \widehat{w} \partial_{\widehat{z}} \widehat{v} - \partial_{\widehat{x}} (\overline{\eta} (\partial_{\widehat{y}} \widehat{u} + \partial_{\widehat{x}} \widehat{v})) \\ &\quad - \partial_{\widehat{y}} (2 \overline{\eta} \partial_{\widehat{y}} \widehat{u}) - \frac{1}{\epsilon^2} \partial_{\widehat{z}} (\overline{\eta} \partial_{\widehat{z}} \widehat{v}) - \partial_{\widehat{z}} (\overline{\eta} \partial_{\widehat{y}} \widehat{w}) + \partial_{\widehat{y}} \widehat{\mathrm{p}} = 0, \end{aligned} \tag{6.19}$$

$$\begin{aligned} &\partial_{\widehat{t}} \widehat{w} + \widehat{u} \partial_{\widehat{x}} \widehat{w} + \widehat{v} \partial_{\widehat{y}} \widehat{w} + \widehat{w} \partial_{\widehat{z}} \widehat{w} - \partial_{\widehat{x}} (\overline{\eta} \partial_{\widehat{x}} \widehat{w}) - \frac{1}{\epsilon^2} \partial_{\widehat{x}} (\overline{\eta} \partial_{\widehat{z}} \widehat{u}) \\ &\quad - \partial_{\widehat{y}} (2 \overline{\eta} \partial_{\widehat{y}} \widehat{w}) - \frac{1}{\epsilon^2} \partial_{\widehat{y}} (\overline{\eta} \partial_{\widehat{z}} \widehat{v}) - \frac{1}{\epsilon^2} \partial_{\widehat{z}} (2 \overline{\eta} \partial_{\widehat{z}} \widehat{w}) + \frac{1}{\epsilon^2} \partial_{\widehat{z}} \widehat{\mathrm{p}} = \frac{1}{\epsilon^2} \overline{\mathrm{g}} \end{aligned} \tag{6.20}$$

and

$$\begin{aligned} &\widehat{c}_v \partial_{\widehat{t}} \widehat{T} + \widehat{c}_v \widehat{u} \partial_{\widehat{x}} \widehat{T} + \widehat{c}_v \widehat{v} \partial_{\widehat{y}} \widehat{T} + \widehat{c}_v \widehat{w} \partial_{\widehat{z}} \widehat{T} - \partial_{\widehat{x}} (\overline{\lambda} \partial_{\widehat{x}} \widehat{T}) \\ &\quad - \partial_{\widehat{y}} (\overline{\lambda} \partial_{\widehat{y}} \widehat{T}) - \frac{1}{\epsilon^2} \partial_{\widehat{z}} (\overline{\lambda} \partial_{\widehat{z}} \widehat{T}) = \overline{\Phi}, \end{aligned} \tag{6.21}$$

where the reduced viscous dissipation $\overline{\Phi}$ is given by

$$\begin{aligned} \overline{\Phi} = \frac{1}{2} \overline{\eta} \mathrm{Ec} \bigg(&4 (\partial_{\widehat{x}} \widehat{u})^2 + 4 (\partial_{\widehat{y}} \widehat{v})^2 + 4 (\partial_{\widehat{z}} \widehat{w})^2 + 2 (\partial_{\widehat{y}} \widehat{u} + \partial_{\widehat{x}} \widehat{v})^2 \\ &+ 2 \left(\frac{1}{\epsilon} \partial_{\widehat{z}} \widehat{v} + \epsilon \partial_{\widehat{y}} \widehat{w} \right)^2 + 2 \left(\epsilon \partial_{\widehat{x}} \widehat{w} + \frac{1}{\epsilon} \partial_{\widehat{z}} \right)^2 \bigg). \end{aligned} \tag{6.22}$$

Remark 6.3. The internal energy per unit mass can also be written $e = e^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_v(\tau)\, d\tau$ and the total energy per unit mass is given by $e^{\mathrm{tot}} = e + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}$. The reduced total energy per unit mass $\hat{e}^{\mathrm{tot}}$ can also be written as $\hat{e}^{\mathrm{tot}} = \hat{e}^{\mathrm{st}} + \int_{\widehat{T}^{\mathrm{st}}}^{\widehat{T}} \widehat{c}_v(\tau) d\tau + \frac{1}{2} \mathrm{Ec} (\widehat{u}^2 + \widehat{v}^2 + \epsilon^2 \widehat{w}^2)$.

6.3. *Rescaled boundary conditions*

Upon using the general notation associated with rescaled variables, the reduced kinematic boundary condition at the top rescaled free boundary $\widehat{z} - \widehat{h}_{\text{gas}} = 0$ can be written as

$$\widehat{w} = \partial_{\widehat{t}}\widehat{h}_{\text{gas}} + \widehat{u}\partial_{\widehat{x}}\widehat{h}_{\text{gas}} + \widehat{v}\partial_{\widehat{y}}\widehat{h}_{\text{gas}},$$

whereas the dynamic condition reads

$$\widehat{\boldsymbol{\sigma}} \cdot \widehat{\mathbf{n}}_{\text{gas}} = \widehat{\boldsymbol{\sigma}}_{\text{gas}} \cdot \widehat{\mathbf{n}}_{\text{gas}} - \epsilon^2 \overline{\gamma}_{\text{f/g}} \widehat{\mathcal{C}}_{\text{gas}} \widehat{\mathbf{n}}_{\text{gas}},$$

where

$$\widehat{\mathbf{n}}_{\text{gas}} = ((\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}})^2 + (\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}})^2 + 1)^{-1/2}(-\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}}, -\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}}, 1)^t,$$

and where the normal component of the stress tensor in gas reads

$$\widehat{\boldsymbol{\sigma}}_{\text{gas}} \cdot \widehat{\mathbf{n}}_{\text{gas}} = -\widehat{\mathrm{p}}_{\text{atm}} \widehat{\mathbf{n}}_{\text{gas}}.$$

By decomposing the dynamic boundary condition componentwise, we obtain the three equations

$$\begin{aligned} &\epsilon\widehat{\mathrm{p}}\partial_{\widehat{x}}\widehat{h}_{\text{gas}} + \widehat{\eta}\left(2\partial_{\widehat{x}}\widehat{u}(-\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}}) + (\partial_{\widehat{y}}\widehat{u} + \partial_{\widehat{x}}\widehat{v})(-\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}}) + \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{u} + \epsilon\partial_{\widehat{x}}\widehat{w}\right) \\ &\quad = \epsilon^3\overline{\gamma}_{\text{f/g}}\widehat{\mathcal{C}}_{\text{gas}}\partial_{\widehat{x}}\widehat{h}_{\text{gas}}, \end{aligned} \tag{6.23}$$

$$\begin{aligned} &\epsilon\widehat{\mathrm{p}}\partial_{\widehat{y}}\widehat{h}_{\text{gas}} + \widehat{\eta}\left((\partial_{\widehat{y}}\widehat{u} + \partial_{\widehat{x}}\widehat{v})(-\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}}) + 2\partial_{\widehat{y}}\widehat{v}(-\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}}) + \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{v} + \epsilon\partial_{\widehat{y}}\widehat{w}\right) \\ &\quad = \epsilon^3\overline{\gamma}_{\text{f/g}}\widehat{\mathcal{C}}_{\text{gas}}\partial_{\widehat{y}}\widehat{h}_{\text{gas}}, \end{aligned} \tag{6.24}$$

$$\begin{aligned} &-\widehat{\mathrm{p}} + \widehat{\eta}\left(\left(\epsilon\partial_{\widehat{x}}\widehat{w} + \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{u}\right)(-\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}}) + \left(\epsilon\partial_{\widehat{y}}\widehat{w} + \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{v}\right)(-\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}}) + 2\partial_{\widehat{z}}\widehat{w}\right) \\ &\quad = -\epsilon^2\overline{\gamma}_{\text{f/g}}\widehat{\mathcal{C}}_{\text{gas}}, \end{aligned} \tag{6.25}$$

where the reduced curvature $\widehat{\mathcal{C}}_{\text{gas}}$ can be written

$$\widehat{\mathcal{C}}_{\text{gas}} = \frac{\partial^2_{\widehat{x}}\widehat{h}_{\text{gas}}(1 + \epsilon^2(\partial_{\widehat{y}}\widehat{h}_{\text{gas}})^2) + \partial^2_{\widehat{y}}\widehat{h}_{\text{gas}}(1 + \epsilon^2(\partial_{\widehat{x}}\widehat{h}_{\text{gas}})^2) - 2\epsilon^2\partial_{\widehat{x}}\widehat{h}_{\text{gas}}\partial_{\widehat{y}}\widehat{h}_{\text{gas}}\partial^2_{\widehat{x}\widehat{y}}\widehat{h}_{\text{gas}}}{(1 + \epsilon^2(\partial_{\widehat{x}}\widehat{h}_{\text{gas}})^2 + \epsilon^2(\partial_{\widehat{y}}\widehat{h}_{\text{gas}})^2)^{3/2}}. \tag{6.26}$$

Similarly, at the reduced free boundary $\widehat{z} - \widehat{h}_{\text{sub}} = 0$ between the fluid and the substrate we can write that

$$\widehat{w} = \partial_{\widehat{t}}\widehat{h}_{\text{sub}} + \widehat{u}\partial_{\widehat{x}}\widehat{h}_{\text{sub}} + \widehat{v}\partial_{\widehat{y}}\widehat{h}_{\text{sub}},$$

and the dynamic condition reads

$$\widehat{\boldsymbol{\sigma}} \cdot \widehat{\mathbf{n}}_{\text{sub}} = \widehat{\boldsymbol{\sigma}}_{\text{sub}} \cdot \widehat{\mathbf{n}}_{\text{sub}} - \epsilon^2 \overline{\gamma}_{\text{f/s}} \widehat{\mathcal{C}}_{\text{sub}} \widehat{\mathbf{n}}_{\text{sub}},$$

with

$$\widehat{\mathbf{n}}_{\text{sub}} = ((\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}})^2 + (\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}})^2 + 1)^{-1/2}(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}, \epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}, -1)^t.$$

The normal component of the stress−tensor in the fluid substrate reads

$$\widehat{\boldsymbol{\sigma}}_{\text{sub}} \cdot \widehat{\mathbf{n}}_{\text{sub}} = -\widehat{\mathrm{p}}_{\text{sub}}\widehat{\mathbf{n}}_{\text{sub}},$$

where $\widehat{\mathrm{p}}_{\text{sub}}$ is assumed to be hydrostatic. The dynamic vector boundary condition can be decomposed componentwise and yields that

$$\begin{aligned} &-\epsilon\widehat{\mathrm{p}}\partial_{\widehat{x}}\widehat{h}_{\text{sub}} + \widehat{\eta}\left(2\partial_{\widehat{x}}\widehat{u}(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}) + (\partial_{\widehat{y}}\widehat{u} + \partial_{\widehat{x}}\widehat{v})(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}) - \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{u} - \epsilon\partial_{\widehat{x}}\widehat{w}\right) \\ &\quad = -\frac{1}{\mathsf{a}}\widehat{h}_{\text{sub}}\,\overline{\mathrm{g}}(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}) - \epsilon^3\,\overline{\gamma}_{\text{f/s}}\widehat{\mathcal{C}}_{\text{sub}}\partial_{\widehat{x}}\widehat{h}_{\text{sub}}, \end{aligned} \tag{6.27}$$

$$\begin{aligned} &-\epsilon\widehat{\mathrm{p}}\partial_{\widehat{y}}\widehat{h}_{\text{sub}} + \widehat{\eta}\left((\partial_{\widehat{y}}\widehat{u} + \partial_{\widehat{x}}\widehat{v})(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}) + 2\partial_{\widehat{y}}\widehat{v}(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}) - \frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{v} - \epsilon\partial_{\widehat{y}}\widehat{w}\right) \\ &\quad = -\frac{1}{\mathsf{a}}\widehat{h}_{\text{sub}}\,\overline{\mathrm{g}}(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}) - \epsilon^3\,\overline{\gamma}_{\text{f/s}}\widehat{\mathcal{C}}_{\text{sub}}\partial_{\widehat{y}}\widehat{h}_{\text{sub}}, \end{aligned} \tag{6.28}$$

$$\begin{aligned} &\widehat{\mathrm{p}} + \widehat{\eta}\left(\left(\frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{u} + \epsilon\partial_{\widehat{x}}\widehat{w}\right)(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}) + \left(\frac{1}{\epsilon}\partial_{\widehat{z}}\widehat{v} + \epsilon\partial_{\widehat{y}}\widehat{w}\right)(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}) - 2\partial_{\widehat{z}}\widehat{w}\right) \\ &\quad = \frac{1}{\mathsf{a}}\widehat{h}_{\text{sub}}\,\overline{\mathrm{g}} + \epsilon^2\,\overline{\gamma}_{\text{f/s}}\widehat{\mathcal{C}}_{\text{sub}}. \end{aligned} \tag{6.29}$$

Finally, the rescaled thermal boundary conditions at the top and bottom interfaces, are of the form

$$-\overline{\lambda}\;\frac{-\partial_{\widehat{x}}\widehat{T}(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}}) - \partial_{\widehat{y}}\widehat{T}(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}}) + 1/\epsilon\,\partial_{\widehat{z}}\widehat{T}}{((\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{gas}})^2 + (\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{gas}})^2 + 1)^{1/2}} = \epsilon\overline{\lambda}^*_{\text{gas}}(\widehat{T} - \widehat{T}_{\text{gas}}),$$

$$-\overline{\lambda}\;\frac{\partial_{\widehat{x}}\widehat{T}(\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}}) + \partial_{\widehat{y}}\widehat{T}(\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}}) - 1/\epsilon\,\partial_{\widehat{z}}\widehat{T}}{((\epsilon\partial_{\widehat{x}}\widehat{h}_{\text{sub}})^2 + (\epsilon\partial_{\widehat{y}}\widehat{h}_{\text{sub}})^2 + 1)^{1/2}} = \epsilon\overline{\lambda}^*_{\text{sub}}(\widehat{T} - \widehat{T}_{\text{sub}}),$$

where $\widehat{T}_{\text{gas}}$ and $\widehat{T}_{\text{sub}}$ are the rescaled given temperatures in gas and in the substrate flow respectively, and where $\overline{\lambda}^*_{\text{gas}}$ and $\overline{\lambda}^*_{\text{sub}}$ are the rescaled heat exchange coefficients.

7. Derivation of the Saint-Venant Equations

The governing equations presented in Sec. 2 and investigated in Secs. 3–5 are now derived from an asymptotic analysis of the three-dimensional incompressible equations modeling thin viscous sheets over fluid substrates presented in Sec. 6.

Asymptotic expansions are a powerful tool for deriving governing equations of multiscale medias. We refer in particular to the monographs of Milton Van Dyke[57] for asymptotic methods in fluid mechanics, Roseau[49] and Sanchez-Hubert and Sanchez-Palencia[51] for asymptotic analysis of vibrating continuous media, and Sanchez-Palencia[50] and Benssousan, Lions and Papanicolaou[4] for asymptotic expansions in homogenization theory. In the context of thin viscous sheets over fluid substrates we mention in particular Howell[31,32] who investigated isothermal flows. Gerbeau and Perthame have revisited the derivation and validated the Saint-Venant model by a direct numerical comparison with the underlying incompressible model.[19] Audusse *et al.* have also recently investigated multilayer media[1,2] and Bresch and Noble have investigated mathematically the situation of nonflat bottoms.[11]

The two-dimensional Saint-Venant system of partial differential equations with an energy equation and temperature-dependent transport coefficient will be obtained as the zeroth-order limit of the three-dimensional incompressible model presented in Sec. 6. We remind that, in the asymptotic limit, the fluid parameters Re, Pr, Ec, $\overline{\mathrm{Fr}}$, $\overline{\mathrm{Ca}}$, and $\overline{\mathrm{Ex}}$ are assumed to be of zeroth-order with respect to ϵ. The quantities associated with the three-dimensional incompressible model are generally denoted by roman fonts whereas the quantities associated with the Saint-Venant two-dimensional asymptotic limit will be denoted with italic fonts. The pressure in the three-dimensional flow is denoted by p for instance whereas it will be denoted by p in the two-dimensional Saint-Venant limit model.

7.1. *Asymptotic expansions*

In order to derive the Saint-Venant equations modeling thin viscous sheets over a fluid substrate from the three-dimensional fluid equations described in Sec. 6, we expand in powers of the small parameter ϵ^2 the fluid variables

$$\widehat{u} = \widehat{u}_0 + \epsilon^2 \widehat{u}_2 + \mathcal{O}(\epsilon^4), \tag{7.1}$$

$$\widehat{v} = \widehat{v}_0 + \epsilon^2 \widehat{v}_2 + \mathcal{O}(\epsilon^4), \tag{7.2}$$

$$\widehat{w} = \widehat{w}_0 + \epsilon^2 \widehat{w}_2 + \mathcal{O}(\epsilon^4), \tag{7.3}$$

$$\widehat{T} = \widehat{T}_0 + \epsilon^2 \widehat{T}_2 + \mathcal{O}(\epsilon^4). \tag{7.4}$$

We also expand the free boundaries h_{gas} and h_{sub} and we define

$$h(t,x,y) = h_{\mathrm{gas}}(t,x,y) - h_{\mathrm{sub}}(t,x,y), \tag{7.5}$$

in such a way that

$$\widehat{h} = \widehat{h}_0 + \epsilon^2 \widehat{h}_2 + \mathcal{O}(\epsilon^4), \tag{7.6}$$

$$\widehat{h}_{\mathrm{gas}} = \widehat{h}_{\mathrm{gas}0} + \epsilon^2 \widehat{h}_{\mathrm{gas}2} + \mathcal{O}(\epsilon^4), \tag{7.7}$$

$$\widehat{h}_{\mathrm{sub}} = \widehat{h}_{\mathrm{sub}0} + \epsilon^2 \widehat{h}_{\mathrm{sub}2} + \mathcal{O}(\epsilon^4). \tag{7.8}$$

Note that, after some algebra, only the factor ϵ^2 appears in the rescaled equations presented in Secs. 6.2–6.3. The asymptotic expansions (7.1)–(7.8) in terms of ϵ^2 are thus natural as they are in the small Mach number limit.[20]

7.2. *Zeroth-order terms and compressibility*

The terms of order ϵ^{-2} in the $\widehat{u}$ and $\widehat{v}$ governing equations first yield that

$$\partial_{\widehat{z}}(\overline{\eta}_0 \partial_{\widehat{z}} \widehat{u}_0) = 0,$$
$$\partial_{\widehat{z}}(\overline{\eta}_0 \partial_{\widehat{z}} \widehat{v}_0) = 0,$$

where

$$\overline{\eta}_0 = \overline{\eta}(\widehat{T}_0).$$

These relations show that $\overline{\eta}_0 \partial_{\widehat{z}} \widehat{u}_0$ and $\overline{\eta}_0 \partial_{\widehat{z}} \widehat{v}_0$ are constants. However, the ϵ^{-1} terms in the dynamic boundary conditions at the fluid/gas and fluid/substrate interfaces yield that $\partial_{\widehat{z}} \widehat{u}_0 = 0$ and $\partial_{\widehat{z}} \widehat{v}_0 = 0$ at both interfaces. We thus deduce that $\partial_{\widehat{z}} \widehat{u}_0 = 0$ and $\partial_{\widehat{z}} \widehat{v}_0 = 0$ for all $\widehat{z}$ in such a way that

$$\widehat{u}_0 = \widehat{u}_0(\widehat{t}, \widehat{x}, \widehat{y}),$$
$$\widehat{v}_0 = \widehat{v}_0(\widehat{t}, \widehat{x}, \widehat{y}).$$

Similarly, the energy conservation equation yields at order ϵ^{-2} that

$$-\partial_{\widehat{z}}(\overline{\lambda}_0 \partial_{\widehat{z}} \widehat{T}_0) = \overline{\eta}_0((\partial_{\widehat{z}} \widehat{u}_0)^2 + (\partial_{\widehat{z}} \widehat{v}_0)^2),$$

where

$$\overline{\lambda}_0 = \overline{\lambda}(\widehat{T}_0),$$

in such a way that $\partial_{\widehat{z}}(\overline{\lambda}_0 \partial_{\widehat{z}} \widehat{T}_0) = 0$ since $\partial_{\widehat{z}} \widehat{u}_0 = 0$ and $\partial_{\widehat{z}} \widehat{v}_0 = 0$. Since the ϵ^{-1} terms of the thermal boundary conditions yield that $\overline{\lambda}_0 \partial_{\widehat{z}} \widehat{T}_0 = 0$ at both the fluid/gas and fluid/substrate interfaces, we again conclude that $\partial_{\widehat{z}} \widehat{T}_0 = 0$ for all $\widehat{z}$ in such a way that

$$\widehat{T}_0 = \widehat{T}_0(\widehat{t}, \widehat{x}, \widehat{y}).$$

This shows that $\widehat{h}_0$, $\widehat{u}_0$, $\widehat{v}_0$, and $\widehat{T}_0$ — and incidentally $\overline{\eta}_0$ and $\overline{\lambda}_0$ — only depend on $(\widehat{t}, \widehat{x}, \widehat{y})$, and $\widehat{h}_0$, $\widehat{u}_0$, $\widehat{v}_0$, and $\widehat{T}_0$ will constitute the variables of the resulting Saint-Venant two-dimensional model. We will also denote by $\widehat{\boldsymbol{v}}_0$ the two-dimensional velocity vector $\widehat{\boldsymbol{v}}_0 = (\widehat{u}_0, \widehat{v}_0)^t$.

On the other hand, from the incompressibility equation at zeroth-order we obtain that

$$\partial_{\widehat{z}} \widehat{w}_0 = -(\partial_{\widehat{x}} \widehat{u}_0 + \partial_{\widehat{y}} \widehat{v}_0),$$

so that $\partial_{\widehat{z}} \widehat{w}_0$ is independent of $\widehat{z}$. This shows that $\widehat{w}_0$ is an affine function of $\widehat{z}$ and that

$$\widehat{w}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas}0}) - \widehat{w}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{sub}0}) = -(\partial_{\widehat{x}} \widehat{u}_0 + \partial_{\widehat{y}} \widehat{v}_0)(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}).$$

From the zeroth-order kinematic conditions at $\widehat{z} = \widehat{h}_{\mathrm{sub}0}$ and $\widehat{z} = \widehat{h}_{\mathrm{sub}0}$ we next deduce that

$$\begin{aligned}\partial_{\widehat{t}}(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}) + \widehat{u}_0 \partial_{\widehat{x}}(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}) + \widehat{v}_0 \partial_{\widehat{y}}(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}) \\ = -(\partial_{\widehat{x}} \widehat{u}_0 + \partial_{\widehat{y}} \widehat{v}_0)(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}),\end{aligned} \tag{7.9}$$

which finally yields that

$$\partial_{\widehat{t}}\,\widehat{h}_0 + \partial_{\widehat{x}}(\widehat{h}_0\widehat{u}_0) + \partial_{\widehat{y}}(\widehat{h}_0\widehat{v}_0) = 0. \tag{7.10}$$

We have thus obtained a compressible model where the zeroth-order height $\widehat{h}_0$ plays the role of a density.

7.3. *Zeroth-order pressure*

From the zeroth-order terms of the normal momentum conservation equation (6.20) we next obtain that

$$-\partial_{\widehat{x}}(\overline{\eta}_0\partial_{\widehat{z}}\widehat{u}_0) - \partial_{\widehat{y}}(\overline{\eta}_0\partial_{\widehat{z}}\widehat{v}_0) - 2\partial_{\widehat{z}}(\overline{\eta}_0\partial_{\widehat{z}}\widehat{w}_0) + \partial_{\widehat{z}}\widehat{\mathrm{p}}_0 = \overline{\mathrm{g}}_0,$$

but since $\partial_{\widehat{z}}\widehat{u}_0 = 0$, $\partial_{\widehat{z}}\widehat{v}_0 = 0$, $\partial_{\widehat{z}}\overline{\eta}_0 = 0$ and $\partial_{\widehat{z}}^2\,\widehat{w}_0 = 0$, we deduce from this relation that

$$\partial_{\widehat{z}}\widehat{\mathrm{p}}_0 = \overline{\mathrm{g}}_0.$$

This shows that the pressure is hydrostatic since

$$\overline{\mathrm{g}}_0 = \overline{\mathrm{g}} = \mathrm{Cte},$$

where $\overline{\boldsymbol{g}} = \overline{\boldsymbol{g}}_0 = (0, 0, \overline{\mathrm{g}}_0)^t$ and $\overline{\mathrm{g}}_0$ is negative. The relation $\partial_{\widehat{z}}\widehat{\mathrm{p}}_0 = \overline{\mathrm{g}}_0$ implies that

$$\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{z}) = \widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas}0}) + \overline{\mathrm{g}}_0(\widehat{z} - \widehat{h}_{\mathrm{gas}0}),$$

but the third component of the dynamic condition at the fluid/gas interface also yields at zeroth-order that $-\widehat{\mathrm{p}}_0 + 2\overline{\eta}_0\partial_{\widehat{z}}\widehat{w}_0 = 0$ at $\widehat{z} = \widehat{h}_{\mathrm{gas}0}$ in such a way that

$$\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{z}) = 2\overline{\eta}_0\partial_{\widehat{z}}\widehat{w}_0 + \overline{\mathrm{g}}_0(\widehat{z} - \widehat{h}_{\mathrm{gas}0}).$$

On the other hand, the dynamic condition at zeroth-order at the fluid/substrate interface gives $-\widehat{\mathrm{p}}_0 + 2\overline{\eta}_0\partial_{\widehat{z}}\widehat{w}_0 = -\frac{1}{\mathsf{a}}\widehat{h}_{\mathrm{sub}0}\,\overline{\mathrm{g}}_0$ at $\widehat{z} = \widehat{h}_{\mathrm{sub}0}$ so that

$$\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas}0}) - \widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{sub}0}) = -\frac{1}{\mathsf{a}}\widehat{h}_{\mathrm{sub}0}\,\overline{\mathrm{g}}_0,$$

and since the pressure at zeroth-order is hydrostatic we also have

$$\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas}0}) - \widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{sub}0}) = \overline{\mathrm{g}}_0(\widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0}).$$

We deduce from these relations that $\widehat{h}_0 = \widehat{h}_{\mathrm{gas}0} - \widehat{h}_{\mathrm{sub}0} = -\frac{1}{\mathsf{a}}\widehat{h}_{\mathrm{sub}0}$ and finally that

$$\widehat{h}_{\mathrm{gas}0} = (1 - \mathsf{a})\widehat{h}_0, \quad \widehat{h}_{\mathrm{sub}0} = -\mathsf{a}\widehat{h}_0. \tag{7.11}$$

These conditions (7.11) are easily interpreted as an equilibrium condition above the substrate bath. Since the height of the outer free substrate bath is taken to be zero, we have of course $\widehat{h}_{\mathrm{gas}0} > 0$ and $\widehat{h}_{\mathrm{sub}0} < 0$. Finally, since $\partial_{\widehat{z}}\widehat{w}_0 = -(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)$ and $\widehat{h}_{\mathrm{gas}0} = (1 - \mathsf{a})\widehat{h}_0$, we have established that

$$\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{z}) = -2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0) + \overline{\mathrm{g}}_0(\widehat{z} - (1 - \mathsf{a})\widehat{h}_0). \tag{7.12}$$

7.4. Zeroth-order momentum equations

The horizontal momentum conservation equations at zeroth-order yield

$$\begin{aligned}\partial_{\widehat{t}}\,\widehat{u}_0 + \widehat{u}_0\partial_{\widehat{x}}\,\widehat{u}_0 + \widehat{v}_0\partial_{\widehat{y}}\widehat{u}_0 - \partial_{\widehat{x}}\,(2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0) - \partial_{\widehat{y}}\,(\overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)) \\ - \partial_{\widehat{z}}(\overline{\eta}_0\partial_{\widehat{z}}\widehat{u}_2) - \partial_{\widehat{z}}(\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{w}_0) + \partial_{\widehat{x}}\,\widehat{\mathrm{p}}_0 = 0\end{aligned} \tag{7.13}$$

and

$$\begin{aligned}\partial_{\widehat{t}}\,\widehat{v}_0 + \widehat{u}_0\partial_{\widehat{x}}\,\widehat{v}_0 + \widehat{v}_0\partial_{\widehat{y}}\widehat{v}_0 - \partial_{\widehat{x}}\,(\overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)) - \partial_{\widehat{y}}\,(2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0) \\ - \partial_{\widehat{z}}(\overline{\eta}_0\partial_{\widehat{z}}\widehat{u}_2) - \partial_{\widehat{z}}(\overline{\eta}_0\partial_{\widehat{y}}\,\widehat{w}_0) + \partial_{\widehat{y}}\widehat{\mathrm{p}}_0 = 0.\end{aligned} \tag{7.14}$$

Integrating the first equation between $\widehat{h}_{\mathrm{sub0}}$ and $\widehat{h}_{\mathrm{gas0}}$ we obtain that

$$\begin{aligned}\widehat{h}_0(\partial_{\widehat{t}}\,\widehat{u}_0 + \widehat{u}_0\partial_{\widehat{x}}\,\widehat{u}_0 + \widehat{v}_0\partial_{\widehat{y}}\widehat{u}_0 - \partial_{\widehat{x}}\,(2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0)) - \widehat{h}_0\partial_{\widehat{y}}\,(\overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)) \\ - \overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{u}_2 + \partial_{\widehat{x}}\,\widehat{w}_0]\!] + \int_{\widehat{h}_{\mathrm{sub0}}}^{\widehat{h}_{\mathrm{gas0}}} \partial_{\widehat{x}}\,\widehat{\mathrm{p}}_0\, d\widehat{z} = 0,\end{aligned} \tag{7.15}$$

where, for any function ϕ of $(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{z})$, the bracket $[\![\phi]\!]$ denotes the corresponding function of $(\widehat{t}, \widehat{x}, \widehat{y})$ defined by

$$[\![\phi]\!](\widehat{t}, \widehat{x}, \widehat{y}) = \phi(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas0}}(\widehat{t}, \widehat{x}, \widehat{y})) - \phi(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{sub0}}(\widehat{t}, \widehat{x}, \widehat{y})). \tag{7.16}$$

We now use the dynamic boundary condition at zeroth-order in the x-direction at both interfaces to get that

$$\begin{aligned}&\widehat{\mathrm{p}}_0\partial_{\widehat{x}}\,\widehat{h}_{\mathrm{gas0}} - 2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0\partial_{\widehat{x}}\,\widehat{h}_{\mathrm{gas0}} - \overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)\partial_{\widehat{y}}\,\widehat{h}_{\mathrm{gas0}} + \overline{\eta}_0\partial_{\widehat{x}}\,\widehat{w}_0 + \overline{\eta}_0\partial_{\widehat{z}}\widehat{u}_2 = 0, \\ &-\widehat{\mathrm{p}}_0\partial_{\widehat{x}}\,\widehat{h}_{\mathrm{sub0}} + 2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0\partial_{\widehat{x}}\,\widehat{h}_{\mathrm{sub0}} + \overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)\partial_{\widehat{y}}\,\widehat{h}_{\mathrm{sub0}} - \overline{\eta}_0\partial_{\widehat{x}}\,\widehat{w}_0 - \overline{\eta}_0\partial_{\widehat{z}}\widehat{u}_2 \\ &\qquad = \frac{\widehat{h}_{\mathrm{sub0}}}{\mathsf{a}}\,\overline{\mathrm{g}}_0\partial_{\widehat{x}}\,\widehat{h}_{\mathrm{sub0}}.\end{aligned}$$

By adding these relations we deduce that

$$\begin{aligned}\overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{u}_2 + \partial_{\widehat{x}}\,\widehat{w}_0]\!] + \tilde{\mathrm{p}}_0\partial_{\widehat{x}}\,\widehat{h}_0 - 2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0\partial_{\widehat{x}}\,\widehat{h}_0 - \overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)\partial_{\widehat{y}}\,\widehat{h}_0 \\ = -\mathsf{a}\widehat{h}_0\,\overline{\mathrm{g}}_0\partial_{\widehat{x}}\,\widehat{h}_0,\end{aligned} \tag{7.17}$$

where

$$\tilde{\mathrm{p}}_0 = (1 - \mathsf{a})\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{gas0}}) + \mathsf{a}\widehat{\mathrm{p}}_0(\widehat{t}, \widehat{x}, \widehat{y}, \widehat{h}_{\mathrm{sub0}}). \tag{7.18}$$

From the expression (7.12) of $\widehat{\mathrm{p}}_0$ we obtain that

$$\tilde{\mathrm{p}}_0 = -2\overline{\eta}_0(\partial_{\widehat{x}}\,\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0) - \overline{\mathrm{g}}_0\mathsf{a}\widehat{h}_0, \tag{7.19}$$

so that

$$\begin{aligned}\overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{u}_2 + \partial_{\widehat{x}}\,\widehat{w}_0]\!] = 2\overline{\eta}_0\partial_{\widehat{x}}\,\widehat{u}_0\partial_{\widehat{x}}\,\widehat{h}_0 + \overline{\eta}_0(\partial_{\widehat{y}}\,\widehat{u}_0 + \partial_{\widehat{x}}\,\widehat{v}_0)\partial_{\widehat{y}}\,\widehat{h}_0 \\ + 2\overline{\eta}_0(\partial_{\widehat{x}}\,\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)\partial_{\widehat{x}}\,\widehat{h}_0.\end{aligned} \tag{7.20}$$

Furthermore, we deduce from (7.12) that

$$\partial_{\widehat{x}}\widehat{\mathrm{p}}_0 = -\partial_{\widehat{x}}(2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)) - (1-\mathsf{a})\partial_{\widehat{x}}\widehat{h}_0, \tag{7.21}$$

and $\partial_{\widehat{x}}\widehat{\mathrm{p}}_0$ is independent of $\widehat{z}$. Combining (7.15), (7.20), (7.21), and since $\partial_{\widehat{x}}\widehat{\mathrm{p}}_0$ is independent of $\widehat{z}$, we finally obtain that

$$\begin{aligned}
&\widehat{h}_0(\partial_{\widehat{t}}\widehat{u}_0 + \widehat{u}_0\partial_{\widehat{x}}\widehat{u}_0 + \widehat{v}_0\partial_{\widehat{y}}\widehat{u}_0 - \partial_{\widehat{x}}(2\overline{\eta}_0\partial_{\widehat{x}}\widehat{u}_0)) - \widehat{h}_0\partial_{\widehat{y}}(\overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0))\\
&\quad - 2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)\partial_{\widehat{x}}\widehat{h}_0 - 2\overline{\eta}_0\partial_{\widehat{x}}\widehat{u}_0\partial_{\widehat{x}}\widehat{h}_0 - \overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)\partial_{\widehat{y}}\widehat{h}_0\\
&\quad + \widehat{h}_0(-\partial_{\widehat{x}}(2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)) - (1-\mathsf{a})\overline{\mathrm{g}}_0\partial_{\widehat{x}}\widehat{h}_0) = 0.
\end{aligned} \tag{7.22}$$

After some algebra this equation can be rewritten in the form

$$\begin{aligned}
&\widehat{h}_0\partial_{\widehat{t}}\widehat{u}_0 + \widehat{h}_0\widehat{u}_0\partial_{\widehat{x}}\widehat{u}_0 + \widehat{h}_0\widehat{v}_0\partial_{\widehat{y}}\widehat{u}_0 - \partial_{\widehat{x}}(\widehat{h}_0 2\overline{\eta}_0\partial_{\widehat{x}}\widehat{u}_0) - \partial_{\widehat{y}}(\widehat{h}_0\overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0))\\
&\quad - \partial_{\widehat{x}}(\widehat{h}_0 2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)) - \frac{1}{2}(1-\mathsf{a})\overline{\mathrm{g}}_0\partial_{\widehat{x}}\widehat{h}_0^2 = 0.
\end{aligned} \tag{7.23}$$

Using the compressibility equation (7.10) and defining the new pressure

$$\widehat{p}_0 = -\frac{1}{2}(1-\mathsf{a})\overline{\mathrm{g}}_0\widehat{h}_0^2 = \frac{1}{2}(1-\mathsf{a})|\overline{\mathrm{g}}_0|\widehat{h}_0^2, \tag{7.24}$$

not to be confused with $\widehat{\mathrm{p}}_0$, and defining the new viscous tensor

$$\overline{\Pi}_{0\widehat{x}\widehat{x}} = -\overline{\eta}_0\widehat{h}_0(4\partial_{\widehat{x}}\widehat{u}_0 + 2\partial_{\widehat{y}}\widehat{v}_0), \quad \overline{\Pi}_{0\widehat{x}\widehat{y}} = -\overline{\eta}_0\widehat{h}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0), \tag{7.25}$$

the equation governing $\widehat{u}_0$ is rewritten in the form

$$\partial_{\widehat{t}}(\widehat{h}_0\widehat{u}_0) + \partial_{\widehat{x}}(\widehat{h}_0\widehat{u}_0^2) + \partial_{\widehat{y}}(\widehat{h}_0\widehat{u}_0\widehat{v}_0) + \partial_{\widehat{x}}\overline{\Pi}_{0\widehat{x}\widehat{x}} + \partial_{\widehat{y}}\overline{\Pi}_{0\widehat{x}\widehat{y}} + \partial_{\widehat{x}}\widehat{p}_0 = 0. \tag{7.26}$$

We can proceed similarly for the second horizontal momentum conservation equation which yields upon integration between $\widehat{h}_{\mathrm{sub}0}$ and $\widehat{h}_{\mathrm{gas}0}$ that

$$\begin{aligned}
&\widehat{h}_0(\partial_{\widehat{t}}\widehat{v}_0 + \widehat{u}_0\partial_{\widehat{x}}\widehat{v}_0 + \widehat{v}_0\partial_{\widehat{y}}\widehat{v}_0 - \widehat{h}_0\partial_{\widehat{x}}(\overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0))) - \partial_{\widehat{y}}(2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0)\\
&\quad - \overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{v}_2 + \partial_{\widehat{y}}\widehat{w}_0]\!] + \int_{\widehat{h}_{\mathrm{sub}0}}^{\widehat{h}_{\mathrm{gas}0}} \partial_{\widehat{y}}\widehat{\mathrm{p}}_0\, d\widehat{z} = 0.
\end{aligned} \tag{7.27}$$

We now use the dynamic boundary condition at zeroth-order in the y-direction at both interfaces to get that

$$\begin{aligned}
&\widehat{p}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{gas}0} - \overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)\partial_{\widehat{x}}\widehat{h}_{\mathrm{gas}0} - 2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{gas}0}\\
&\qquad + \overline{\eta}_0\partial_{\widehat{y}}\widehat{w}_0 + \overline{\eta}_0\partial_{\widehat{z}}\widehat{v}_2 = 0,\\
&-\widehat{p}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{sub}0} + \overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)\partial_{\widehat{x}}\widehat{h}_{\mathrm{sub}0} + 2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{sub}0}\\
&\qquad - \overline{\eta}_0\partial_{\widehat{y}}\widehat{w}_0 - \overline{\eta}_0\partial_{\widehat{z}}\widehat{v}_2 = -\frac{\widehat{h}_{\mathrm{sub}0}}{\mathsf{a}}\overline{\mathrm{g}}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{sub}0}.
\end{aligned}$$

By adding these relations we deduce that

$$\begin{aligned}\overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{v}_2 + \partial_{\widehat{y}}\widehat{w}_0]\!] + \widetilde{\mathrm{p}}_0\partial_{\widehat{y}}\widehat{h}_0 - \overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)\partial_{\widehat{x}}\widehat{h}_0 - 2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0\partial_{\widehat{y}}\widehat{h}_0 \\ = -\mathsf{a}\widehat{h}_0\,\overline{\mathrm{g}}_0\partial_{\widehat{y}}\widehat{h}_0,\end{aligned} \tag{7.28}$$

so that from (7.18)

$$\begin{aligned}\overline{\eta}_0[\![\partial_{\widehat{z}}\widehat{v}_2 + \partial_{\widehat{y}}\widehat{w}_0]\!] = \overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)\partial_{\widehat{x}}\widehat{h}_0 \\ + 2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0\partial_{\widehat{y}}\widehat{h}_0 + 2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0).\end{aligned} \tag{7.29}$$

Furthermore, we deduce from (7.12) that

$$\partial_{\widehat{y}}\widehat{\mathrm{p}}_0 = -\partial_{\widehat{y}}(2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)) - (1-\mathsf{a})\partial_{\widehat{y}}\widehat{h}_0, \tag{7.30}$$

and $\partial_{\widehat{y}}\widehat{\mathrm{p}}_0$ is independent of $\widehat{z}$. Combining (7.27), (7.29), (7.30), and since $\partial_{\widehat{y}}\widehat{\mathrm{p}}_0$ is independent of $\widehat{z}$, we finally obtain that

$$\begin{aligned}\widehat{h}_0\partial_{\widehat{t}}\widehat{v}_0 + \widehat{h}_0\widehat{u}_0\partial_{\widehat{x}}\widehat{v}_0 + \widehat{h}_0\widehat{v}_0\partial_{\widehat{y}}\widehat{v}_0 - \partial_{\widehat{x}}(\widehat{h}_0\overline{\eta}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)) - \partial_{\widehat{y}}(\widehat{h}_0 2\overline{\eta}_0\partial_{\widehat{y}}\widehat{v}_0) \\ - \partial_{\widehat{y}}(\widehat{h}_0 2\overline{\eta}_0(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)) - \frac{1}{2}(1-\mathsf{a})\overline{\mathrm{g}}_0\partial_{\widehat{y}}\widehat{h}_0^2 = 0.\end{aligned} \tag{7.31}$$

Defining

$$\overline{\mathit{\Pi}}_{0\widehat{y}\widehat{x}} = -\overline{\eta}_0\widehat{h}_0(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0), \quad \overline{\mathit{\Pi}}_{0\widehat{y}\widehat{y}} = -\overline{\eta}_0\widehat{h}_0(2\partial_{\widehat{x}}\widehat{u}_0 + 4\partial_{\widehat{y}}\widehat{v}_0), \tag{7.32}$$

the equation governing $\widehat{v}_0$ is easily rewritten in the form

$$\partial_{\widehat{t}}(\widehat{h}_0\widehat{v}_0) + \partial_{\widehat{x}}(\widehat{h}_0\widehat{u}_0\widehat{v}_0) + \partial_{\widehat{y}}(\widehat{h}_0\widehat{v}_0^2) + \partial_{\widehat{x}}\overline{\mathit{\Pi}}_{0\widehat{y}\widehat{x}} + \partial_{\widehat{y}}\overline{\mathit{\Pi}}_{0\widehat{y}\widehat{y}} + \partial_{\widehat{y}}\widehat{p}_0 = 0. \tag{7.33}$$

Upon defining $\widehat{\boldsymbol{v}}_0 = (\widehat{u}_0, \widehat{v}_0)^t$, $\widehat{\boldsymbol{x}} = (\widehat{x}, \widehat{y})^t$, and

$$\overline{\boldsymbol{\mathit{\Pi}}}_0 = \begin{pmatrix} \overline{\mathit{\Pi}}_{0\widehat{x}\widehat{x}} & \overline{\mathit{\Pi}}_{0\widehat{x}\widehat{y}} \\ \overline{\mathit{\Pi}}_{0\widehat{y}\widehat{x}} & \overline{\mathit{\Pi}}_{0\widehat{y}\widehat{y}} \end{pmatrix}, \tag{7.34}$$

both momentum equations can be rewritten in vector form

$$\partial_{\widehat{t}}(\widehat{h}_0\widehat{\boldsymbol{v}}_0) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\cdot(\widehat{h}_0\widehat{\boldsymbol{v}}_0\otimes\widehat{\boldsymbol{v}}_0 + \widehat{p}_0\boldsymbol{I}) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\cdot\overline{\boldsymbol{\mathit{\Pi}}}_0 = 0, \tag{7.35}$$

in such a way that the height $\widehat{h}_0$ plays the role of a density and $\widehat{p}_0$ the role of a pressure for the two-dimensional Saint-Venant model.

7.5. *Zeroth-order energy equation*

Upon using $\partial_{\widehat{z}}\widehat{u}_0 = \partial_{\widehat{z}}\widehat{v}_0 = \partial_{\widehat{z}}\widehat{T}_0 = \partial_{\widehat{z}}\overline{\lambda}_0 = 0$, the energy conservation equation at zeroth-order yields that

$$\begin{aligned}\widehat{c}_{v0}\partial_{\widehat{t}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{u}_0\partial_{\widehat{x}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{v}_0\partial_{\widehat{y}}\widehat{T}_0 - \partial_{\widehat{x}}(\overline{\lambda}_0\partial_{\widehat{x}}\widehat{T}_0) \\ - \partial_{\widehat{y}}(\overline{\lambda}_0\partial_{\widehat{y}}\widehat{T}_0) - \partial_{\widehat{z}}(\overline{\lambda}_0\partial_{\widehat{z}}\widehat{T}_2) = \overline{\Phi}_0,\end{aligned} \tag{7.36}$$

where the zeroth-order viscous dissipation $\overline{\Phi}_0$ is given by

$$\overline{\Phi}_0 = \frac{1}{2}\,\overline{\eta}_0 \mathrm{Ec}(4(\partial_{\widehat{x}}\widehat{u}_0)^2 + 4(\partial_{\widehat{y}}\widehat{v}_0)^2 + 4(\partial_{\widehat{x}}\widehat{u}_0 + \partial_{\widehat{y}}\widehat{v}_0)^2 + 2(\partial_{\widehat{y}}\widehat{u}_0 + \partial_{\widehat{x}}\widehat{v}_0)^2). \quad (7.37)$$

Integrating Eq. (7.36) between $\widehat{h}_{\mathrm{sub}0}$ and $\widehat{h}_{\mathrm{gas}0}$ we obtain that

$$\begin{aligned}\widehat{h}_0(\widehat{c}_{v0}\partial_{\widehat{t}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{u}_0\partial_{\widehat{x}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{v}_0\partial_{\widehat{y}}\widehat{T}_0 - \partial_{\widehat{x}}(\overline{\lambda}_0\partial_{\widehat{x}}\widehat{T}_0)) - \widehat{h}_0\partial_{\widehat{y}}(\overline{\lambda}_0\widehat{T}_0)\\ - \widehat{h}_0\overline{\Phi}_0 - \overline{\lambda}_0[\![\partial_{\widehat{z}}\widehat{T}_2]\!] = 0.\end{aligned} \quad (7.38)$$

We now use the thermal boundary condition at zeroth-order at both interfaces to get that

$$\begin{aligned}-\overline{\lambda}_0(-\partial_{\widehat{x}}\widehat{T}_0\partial_{\widehat{x}}\widehat{h}_{\mathrm{gas}0} - \partial_{\widehat{y}}\widehat{T}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{gas}0} + \partial_{\widehat{z}}\widehat{T}_2) &= \overline{\lambda}^*_{0\mathrm{gas}}(\widehat{T}_0 - \widehat{T}_{0\mathrm{gas}}),\\ -\overline{\lambda}_0(\partial_{\widehat{x}}\widehat{T}_0\partial_{\widehat{x}}\widehat{h}_{\mathrm{sub}0} + \partial_{\widehat{y}}\widehat{T}_0\partial_{\widehat{y}}\widehat{h}_{\mathrm{sub}0} - \partial_{\widehat{z}}\widehat{T}_2) &= \overline{\lambda}^*_{0\mathrm{sub}}(\widehat{T}_0 - \widehat{T}_{0\mathrm{sub}}).\end{aligned}$$

By adding these equations we obtain that

$$-\overline{\lambda}_0[\![\partial_{\widehat{z}}\widehat{T}_2]\!] + \overline{\lambda}_0\partial_{\widehat{x}}\widehat{T}_0\partial_{\widehat{x}}\widehat{h}_0 + \overline{\lambda}_0\partial_{\widehat{y}}\widehat{T}_0\partial_{\widehat{y}}\widehat{h}_0 = \overline{\lambda}^*_0(\widehat{T}_0 - \widehat{T}_{0\mathrm{mix}}), \quad (7.39)$$

where we have defined

$$\widehat{T}_{0\mathrm{mix}} = \frac{\overline{\lambda}^*_{0\mathrm{gas}}\widehat{T}_{0\mathrm{gas}} + \overline{\lambda}^*_{0\mathrm{sub}}\widehat{T}_{0\mathrm{sub}}}{\overline{\lambda}^*_{0\mathrm{gas}} + \overline{\lambda}^*_{0\mathrm{sub}}}$$

and

$$\overline{\lambda}^*_0 = \overline{\lambda}^*_{0\mathrm{gas}} + \overline{\lambda}^*_{0\mathrm{sub}}.$$

Combining (7.38), (7.39) we obtain that

$$\begin{aligned}\widehat{h}_0(\widehat{c}_{v0}\partial_{\widehat{t}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{u}_0\partial_{\widehat{x}}\widehat{T}_0 + \widehat{c}_{v0}\widehat{v}_0\partial_{\widehat{y}}\widehat{T}_0) - \partial_{\widehat{x}}(\widehat{h}_0\overline{\lambda}_0\partial_{\widehat{x}}\widehat{T}_0)) - \partial_{\widehat{y}}(\widehat{h}_0\overline{\lambda}_0\widehat{T}_0) - \widehat{h}_0\overline{\Phi}_0\\ = -\overline{\lambda}^*_0(\widehat{T}_0 - \widehat{T}_{0\mathrm{mix}}).\end{aligned} \quad (7.40)$$

Furthemore, the dissipation term $\widehat{h}_0\overline{\Phi}_0$ is easily rewritten in the form

$$\widehat{h}_0\overline{\Phi}_0 = -\mathrm{Ec}\,\overline{\boldsymbol{\Pi}}_0 : \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{v}}_0. \quad (7.41)$$

Denoting the heat loss term by

$$\widehat{\mathcal{H}}_0 = -\overline{\lambda}^*_0(\widehat{T}_0 - \widehat{T}_{0\mathrm{mix}}), \quad (7.42)$$

the internal energy per unit mass by

$$\hat{e}_0 = \hat{e}^{\mathrm{st}} + \int_{\widehat{T}^{\mathrm{st}}}^{\widehat{T}_0} \widehat{c}_{v0}(\hat{\tau})d\hat{\tau} + \frac{1}{2}(1-\mathsf{a})\mathrm{Ec}|\overline{\mathrm{g}}_0|\widehat{h}_0, \quad (7.43)$$

and the heat flux vector by

$$\overline{\boldsymbol{Q}}_0 = (\overline{\mathcal{Q}}_{0\widehat{x}}, \overline{\mathcal{Q}}_{0\widehat{y}})^t, \quad (7.44)$$

where

$$\overline{\mathcal{Q}}_{0\widehat{x}} = -\widehat{h}_0\overline{\lambda}_0\partial_{\widehat{x}}\widehat{T}_0, \quad \overline{\mathcal{Q}}_{0\widehat{y}} = -\widehat{h}_0\overline{\lambda}_0\partial_{\widehat{y}}\widehat{T}_0, \tag{7.45}$$

the energy conservation equation is rewritten in the form

$$\widehat{h}_0\partial_{\widehat{t}}\,\widehat{e}_0 + \widehat{h}_0\widehat{\boldsymbol{v}}_0\boldsymbol{\cdot}\boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\,\widehat{e}_0 + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}\overline{\boldsymbol{Q}}_0 + \mathrm{Ec}\widehat{p}_0\boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}\widehat{\boldsymbol{v}}_0 + \mathrm{Ec}\,\overline{\boldsymbol{\Pi}}_0 : \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\widehat{\boldsymbol{v}}_0 = \widehat{\mathcal{H}}_0. \tag{7.46}$$

Note that the inclusion of the term $\frac{1}{2}(1-\mathsf{a})\mathrm{Ec}|\overline{\mathrm{g}}_0|\widehat{h}_0$ in the internal energy is associated with the pressure work term $\mathrm{Ec}\widehat{p}_0\boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}\widehat{\boldsymbol{v}}_0$ in the energy equation thanks to the relation $\partial_{\widehat{t}}\,\widehat{h}_0 + \widehat{\boldsymbol{v}}_0\boldsymbol{\cdot}\boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\widehat{h}_0 + \widehat{h}_0\boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}\widehat{\boldsymbol{v}}_0 = 0$. Finally, upon multiplying the momentum governing equation by the velocity vector $\widehat{\boldsymbol{v}}_0$, we obtain the kinetic energy governing equation, which can be multiplied by Ec and added to the internal energy governing equation in order to obtain the total energy conservation equation in the form

$$\partial_{\widehat{t}}\,(\widehat{h}_0\,\widehat{e}_0^{\,\mathrm{tot}}) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}((\widehat{h}_0\,\widehat{e}_0^{\,\mathrm{tot}} + \mathrm{Ec}\widehat{p}_0)\widehat{\boldsymbol{v}}_0) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}(\overline{\boldsymbol{Q}}_0 + \mathrm{Ec}\,\overline{\boldsymbol{\Pi}}_0\boldsymbol{\cdot}\widehat{\boldsymbol{v}}_0) = \widehat{\mathcal{H}}_0, \tag{7.47}$$

where

$$\widehat{e}_0^{\,\mathrm{tot}} = \widehat{e}_0 + \frac{1}{2}\mathrm{Ec}(\widehat{u}_0^2 + \widehat{v}_0^2), \tag{7.48}$$

is the reduced total energy per unit mass.

7.6. *Resulting model*

From the previous sections, we can summarize the zeroth-order rescaled governing equations in the form

$$\partial_{\widehat{t}}\,\widehat{h}_0 + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}(\widehat{h}_0\widehat{\boldsymbol{v}}_0) = 0, \tag{7.49}$$

$$\partial_{\widehat{t}}\,(\widehat{h}_0\widehat{\boldsymbol{v}}_0) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}(\widehat{h}_0\widehat{\boldsymbol{v}}_0\otimes\widehat{\boldsymbol{v}}_0 + \widehat{p}_0\boldsymbol{I}) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}\overline{\boldsymbol{\Pi}}_0 = 0, \tag{7.50}$$

$$\partial_{\widehat{t}}\,(\widehat{h}_0\,\widehat{e}_0^{\,\mathrm{tot}}) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}((\widehat{h}_0\,\widehat{e}_0^{\,\mathrm{tot}} + \mathrm{Ec}\widehat{p}_0)\widehat{\boldsymbol{v}}_0) + \boldsymbol{\partial}_{\widehat{\boldsymbol{x}}}\boldsymbol{\cdot}(\overline{\boldsymbol{Q}}_0 + \mathrm{Ec}\,\overline{\boldsymbol{\Pi}}_0\boldsymbol{\cdot}\widehat{\boldsymbol{v}}_0) = \widehat{\mathcal{H}}_0. \tag{7.51}$$

The pressure $\widehat{p}_0$ is given by (7.24), the total energy $\widehat{e}_0^{\,\mathrm{tot}}$ by (7.48), and internal energy $\widehat{e}_0$ by (7.43). The viscous tensor $\overline{\boldsymbol{\Pi}}_0$ is given by (7.25), (7.32), the heat flux $\overline{\boldsymbol{Q}}_0$ by (7.45) and the heat loss term $\widehat{\mathcal{H}}_0$ by (7.42).

Upon restoring the physical dimensions of the flow quantities $t = \langle x\rangle\widehat{t}/\langle u\rangle$, $\boldsymbol{x} = \langle x\rangle\widehat{\boldsymbol{x}}$, $h_0 = \epsilon\langle x\rangle\widehat{h}_0$, $\boldsymbol{v}_0 = \langle u\rangle\widehat{\boldsymbol{v}}_0$, $p_0 = \langle h\rangle\langle u\rangle^2\widehat{p}_0$, $e_0^{\mathrm{tot}} = \langle c_v\rangle\langle T\rangle\,\widehat{e}_0^{\,\mathrm{tot}}$, $\boldsymbol{\Pi}_0 = \langle h\rangle\langle u\rangle^2\overline{\boldsymbol{\Pi}}_0$, $\boldsymbol{Q}_0 = \langle h\rangle\langle u\rangle\langle c_v\rangle\langle T\rangle\overline{\boldsymbol{Q}}_0$, and $\mathcal{H}_0 = \langle h\rangle\langle u\rangle\langle c_v\rangle\langle T\rangle\widehat{\mathcal{H}}_0/\langle x\rangle$, we obtain after some algebra that

$$\partial_t h_0 + \boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{\cdot}(h_0\boldsymbol{v}_0) = 0, \tag{7.52}$$

$$\partial_t(h_0\boldsymbol{v}_0) + \boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{\cdot}(h_0\boldsymbol{v}_0\otimes\boldsymbol{v}_0 + p_0\boldsymbol{I}) + \boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{\cdot}\boldsymbol{\Pi}_0 = 0, \tag{7.53}$$

$$\partial_t(h_0 e_0^{\mathrm{tot}}) + \boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{\cdot}((h_0 e_0^{\mathrm{tot}} + p_0)\boldsymbol{v}_0) + \boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{\cdot}(\boldsymbol{Q}_0 + \boldsymbol{\Pi}_0\boldsymbol{\cdot}\boldsymbol{v}_0) = \mathcal{H}_0. \tag{7.54}$$

The scaled thermodynamic relations are

$$e_0^{\text{tot}} = e_0 + \frac{1}{2}(u_0^2 + v_0^2), \tag{7.55}$$

$$e_0 = e_0^{\text{st}} + \int_{T_0^{\text{st}}}^{T_0} c_v(\tau)d\tau + \frac{1}{2}(1-\mathsf{a})|\mathrm{g}_0|h_0, \tag{7.56}$$

$$p_0 = \frac{1}{2}(1-\mathsf{a})|\mathrm{g}_0|h_0^2. \tag{7.57}$$

The scaled viscous tensor is given by

$$\boldsymbol{\Pi}_0 = -\nu_0 h_0(\boldsymbol{\partial_x} \boldsymbol{v}_0 + \boldsymbol{\partial_x} \boldsymbol{v}_0^t + 2\boldsymbol{\partial_x} \cdot \boldsymbol{v}_0 \boldsymbol{I}) = \begin{pmatrix} \Pi_{0xx} & \Pi_{0xy} \\ \Pi_{0yx} & \Pi_{0yy} \end{pmatrix}, \tag{7.58}$$

with

$$\begin{aligned} \Pi_{0xx} &= -\nu_0 h_0(4\partial_x u_0 + 2\partial_y v_0), & \Pi_{0xy} &= -\nu_0 h_0(\partial_y u_0 + \partial_x v_0), \\ \Pi_{0yx} &= -\nu_0 h_0(\partial_y u_0 + \partial_x v_0), & \Pi_{0xy} &= -\nu_0 h_0(2\partial_x u_0 + 4\partial_y v_0), \end{aligned}$$

where the kinematic viscosity is given by

$$\nu_0 = \frac{\eta(T_0)}{\rho}. \tag{7.59}$$

Strictly speaking, only the quantity $\rho\boldsymbol{\Pi}_0/h_0$ is homogeneous to a viscous tensor and $\boldsymbol{\Pi}_0/h_0$ to a kinematic viscous tensor. Similarly, only the quantity $\rho p_0/h_0$ is homogeneous to a pressure and p_0/h_0 to a kinematic pressure. However, the multiplication by h_0 is natural in a two-dimensional context since then internal constraints arise through lines and not surfaces. Finally, the heat flux is given by

$$\boldsymbol{Q}_0 = (\mathcal{Q}_{0x}, \mathcal{Q}_{0y})^t = -\varkappa_0 h_0(\partial_x T_0, \partial_y T_0)^t, \tag{7.60}$$

where the kinematic thermal conductivity is given by

$$\varkappa_0 = \frac{\lambda(T_0)}{\rho}. \tag{7.61}$$

Strictly speaking, only the quantity $\rho\boldsymbol{Q}_0/h_0$ is homogeneous to a heat flux and $\boldsymbol{Q}_0/h_0$ to a kinematic heat flux. On the other hand, the heat loss term reads

$$\mathcal{H}_0 = -\frac{\lambda_0^*}{\rho}(T_0 - T_{0\text{mix}}), \tag{7.62}$$

where $\lambda_0^* = \lambda_{0\text{gas}}^* + \lambda_{0\text{sub}}^*$ and

$$T_{0\text{mix}} = \frac{\lambda_{0\text{gas}}^* T_{0\text{gas}} + \lambda_{0\text{sub}}^* T_{0\text{sub}}}{\lambda_{0\text{gas}}^* + \lambda_{0\text{sub}}^*}. \tag{7.63}$$

This model (7.52)–(7.63) is exactly the model that we have investigated in Secs. 2–5 of this paper.

Remark 7.1. The expression of the viscous tensor (7.58) indicate that there is *always* a volume viscosity term for this viscous sheet over fluid substrates as for polyatomic gases.[7]

Remark 7.2. Saint-Venant models with a local energy partial differential equation should not be confused with isothermal models incorporating a global kinetic energy balance as investigated for instance by Kanayama.[34]

7.7. *Boundary conditions*

We present in this section typical free boundary conditions associated with thin viscous sheets over fluid substrates. These boundary conditions are written at the free boundary of the two-dimensional Saint-Venant model. These boundary conditions are not used in this paper and are only written here for completeness. We also discuss their validity associated with the positivity of the sheet thickness h in the framework of wettability theory.[15]

The two-dimensional Saint-Venant equations governing thin viscous sheets have been derived in the previous sections from the three-dimensional incompressible Navier–Stokes equations governing incompressible fluids. Similarly, the boundary conditions associated with the two-dimensional Saint-Venant model will be derived from the boundary conditions and conservation equations of the three-dimensional model.

Exchanging eventually the role of x and y, we may assume that the free boundary can locally be written in the form $x = X_{\mathrm{b}}(t, y)$. The local geometry of such a free boundary $x = X_{\mathrm{b}}(t, y)$ is depicted in Fig. 2. The boundary conditions associated with the two-dimensional Saint-Venant model at the free boundary $x = X_{\mathrm{b}}(t, y)$ can be decomposed into a kinematic condition, a dynamic momentum boundary condition, and a thermal boundary condition.

We first investigate the dynamic momentum boundary condition at the free boundary. To this aim, we consider a slice of the free boundary of the three-dimensional incompressible model in the plane spanned by $\mathbf{n}_{\mathrm{b}}$ and $\mathbf{e}_z$, where $\mathbf{n}_{\mathrm{b}}$ is the outward unit

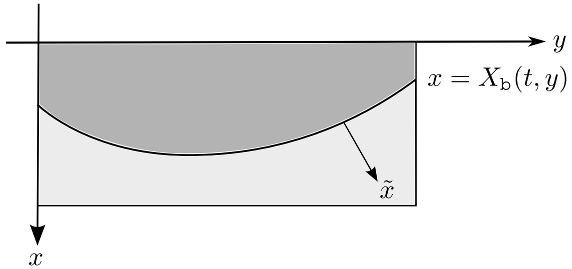

Fig. 2. A local chart of the free boundary in the xy plane.

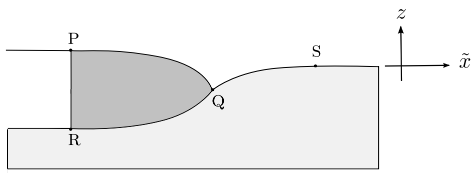

Fig. 3. Schematic of a slice of the thin viscous sheet free boundary.

normal of the free boundary $x = X_{\rm b}$ in the xy plane as depicted in Fig. 2

$$\mathbf{n}_{\rm b} = (1 + (\partial_y X_{\rm b})^2)^{-1/2}(1, -\partial_y X_{\rm b}, 0))^t. \tag{7.64}$$

We next define $\mathbf{e}_{\tilde{x}} = \mathbf{n}_{\rm b}$ and $\mathbf{e}_{\tilde{y}} = \mathbf{e}_z \wedge \mathbf{e}_{\tilde{x}}$ in such a way that $\mathbf{e}_{\tilde{x}}, \mathbf{e}_{\tilde{y}}, \mathbf{e}_z$ form a direct orthonormal basis, and we denote by $(\tilde{x}, \tilde{y}, z)$ the corresponding coordinates so that $\tilde{x}$ is measured along $\mathbf{n}_{\rm b}$. The geometry of the corresponding slice in the plane $(\tilde{x}, z)$ of the free boundary associated with the three-dimensional model is presented in Fig. 3 where the fluid lay above the substrate. The asymptotic dynamic boundary condition is obtained upon integrating the horizontal momentum equation in the domain PQR. Since this domain is assumed to be of size $\mathcal{O}(\epsilon^2)$, all inertial terms will be neglected in comparison with the force terms that are $\mathcal{O}(\epsilon)$. The forces acting on this volume are the surface tension forces, the viscous constraints on PR, and Archimedes' forces on QR. Note that we only consider the horizontal momentum equation so that there is no gravity term.

Since the pressure in the substrate fluid is hydrostatic, and keeping in mind that all pressures are evaluated relative to the atmospheric pressure, the resultant of Archimedes' forces on QR can be written

$$-\int_{\rm QR} \rho_{\rm sub} \mathrm{g} h_{\rm sub} \mathbf{n}_{\rm sub} d\tilde{s}_{\rm sub},$$

where $\tilde{s}_{\rm sub}$ is the arclength along the curve $(\tilde{x}, \tilde{h}_{\rm sub}(t, \tilde{x}, \tilde{y}))$. In this expression, we have denoted $\tilde{h}_{\rm sub}(t, \tilde{x}, \tilde{y}) = h_{\rm sub}(t, x, y)$ and $\mathbf{n}_{\rm sub}$ the corresponding outward oriented normal vector

$$\mathbf{n}_{\rm sub} = \frac{\partial_{\tilde{x}} h_{\rm sub} \mathbf{n}_{\rm b} - \mathbf{e}_z}{\sqrt{1 + (\partial_{\tilde{x}} h_{\rm sub})^2}}.$$

Since the normal vector $\mathbf{n}_{\rm sub}$ is oriented downward, the arc QR must be oriented from Q to R. The horizontal projection of this Archimedes' force is easily evaluated as

$$\int_{\rm QR} \rho_{\rm sub} \mathrm{g} h_{\rm sub} \mathbf{n}_{\rm sub} \cdot \mathbf{n}_{\rm b} d\tilde{s}_{\rm sub} = \frac{1}{2} \rho_{\rm sub} \mathrm{g} (h_{\rm sub}^2(\mathrm{Q}) - h_{\rm sub}^2(\mathrm{R})), \tag{7.65}$$

since

$$\mathbf{n}_{\rm sub} \cdot \mathbf{n}_{\rm b} = \frac{\partial_{\tilde{x}} h_{\rm sub}}{\sqrt{1 + (\partial_{\tilde{x}} h_{\rm sub})^2}}, \quad d\tilde{s}_{\rm sub} = -\sqrt{1 + (\partial_{\tilde{x}} h_{\rm sub})^2}\, d\tilde{x}.$$

On the other hand, the curvatures in the $\tilde{x}$-direction are $\mathcal{O}(1/\epsilon)$ whereas the curvatures in the $\tilde{y}$-direction are $\mathcal{O}(\epsilon)$ — and may be neglected — in such a way that the total curvatures $\mathcal{C}_{\rm gas}$ and $\mathcal{C}_{\rm sub}$ may be approximated as

$$\mathcal{C}_{\rm gas} = \frac{\partial^2_{\tilde{x}} h_{\rm gas}}{(1+(\partial_{\tilde{x}} h_{\rm gas})^2)^{3/2}}, \quad \mathcal{C}_{\rm sub} = \frac{\partial^2_{\tilde{x}} h_{\rm sub}}{(1+(\partial_{\tilde{x}} h_{\rm sub})^2)^{3/2}}. \tag{7.66}$$

Using these expressions, the surface tension forces acting on PQR can be written

$$\int_{\rm PQ} \partial_{\tilde{s}}(\gamma_{\rm f/g}\boldsymbol{\tau}_{\rm gas})d\tilde{s}_{\rm gas} + \int_{\rm QR} \partial_{\tilde{s}}(\gamma_{\rm f/s}\boldsymbol{\tau}_{\rm sub})d\tilde{s}_{\rm sub},$$

where $\boldsymbol{\tau}_{\rm sub}$ is the tangent vector along the arc $(\tilde{x}, \tilde{h}_{\rm sub}(t,\tilde{x},\tilde{y}))$ oriented from Q to R, and $\tilde{s}_{\rm gas}$ and $\boldsymbol{\tau}_{\rm gas}$ are the arclength and tangent vector along the arc $(\tilde{x}, \tilde{h}_{\rm gas}(t,\tilde{x},\tilde{y}))$ oriented from P to Q. We have used in particular the differential relations $\partial_{\tilde{s}}\boldsymbol{\tau}_{\rm gas} = \mathcal{C}_{\rm gas}\mathbf{n}_{\rm gas}$ and $\partial_{\tilde{s}}\boldsymbol{\tau}_{\rm sub} = \mathcal{C}_{\rm sub}\mathbf{n}_{\rm sub}$. Furthermore, since $\mathbf{n}_{\rm gas}$ is upward and the arc PQ oriented from P to Q, we may assume that at zeroth-order $\boldsymbol{\tau}_{\rm gas}(\mathrm{P}) = \boldsymbol{e}_{\tilde{x}} = \mathbf{n}_{\rm b}$. Similarly, since $\mathbf{n}_{\rm sub}$ is downward and the arc QR oriented from Q to R, we may assume that at zeroth-order $\boldsymbol{\tau}_{\rm sub}(\mathrm{R}) = -\boldsymbol{e}_{\tilde{x}} = -\mathbf{n}_{\rm b}$. Integrating along the arcs PQ and QR, the surface tension forces are thus found to be

$$\gamma_{\rm f/s}(\boldsymbol{\tau}_{\rm sub}(\mathrm{R}) - \boldsymbol{\tau}_{\rm sub}(\mathrm{Q})) + \gamma_{\rm f/g}(\boldsymbol{\tau}_{\rm gas}(\mathrm{Q}) - \boldsymbol{\tau}_{\rm gas}(\mathrm{P})).$$

We now use the fundamental relation relating the tangent vectors at the triple-point Q

$$-\gamma_{\rm f/s}\boldsymbol{\tau}_{\rm sub}(\mathrm{Q}) + \gamma_{\rm f/g}\boldsymbol{\tau}_{\rm gas}(\mathrm{Q}) - \gamma_{\rm g/s}\boldsymbol{\tau}_{\rm ext}(\mathrm{Q}) = 0, \tag{7.67}$$

where $z - h_{\rm ext} = 0$ denotes the free surface between gas and the fluid substrate, $\gamma_{\rm g/s}$ the surface tension between gas and the fluid substrate, S a point as depicted in Fig. 3, $(\tilde{x}, \tilde{h}_{\rm ext}(t,\tilde{x},\tilde{y}))$ the arc QS oriented from Q to S, and $\boldsymbol{\tau}_{\rm ext}$ the corresponding tangent vector. Since this arc is oriented from Q to S with $\mathbf{n}_{\rm ext}$ oriented upward, we may assume that at zeroth-order $\boldsymbol{\tau}_{\rm ext}(\mathrm{S}) = \boldsymbol{e}_{\tilde{x}} = \mathbf{n}_{\rm b}$. This relation (7.67) can be used to simplify the expression of the surface tension forces by eliminating all quantities associated with the triple point Q, provided we can express the tangent vector $\boldsymbol{\tau}_{\rm ext}(\mathrm{Q})$. To this aim, we can use the dynamic equilibrium condition at the gas/substrate interface which states that

$$\boldsymbol{\sigma}_{\rm sub}\cdot\mathbf{n}_{\rm ext} = \boldsymbol{\sigma}_{\rm gas}\cdot\mathbf{n}_{\rm ext} - \mathcal{C}_{\rm ext}\gamma_{\rm g/s}\mathbf{n}_{\rm ext} = \boldsymbol{\sigma}_{\rm gas}\cdot\mathbf{n}_{\rm ext} - \partial_{\tilde{s}}(\gamma_{\rm g/s}\boldsymbol{\tau}_{\rm ext}), \tag{7.68}$$

where $\mathcal{C}_{\rm ext}$ is the total curvature of the surface $z = h_{\rm ext}$ which may also be approximated as

$$\mathcal{C}_{\rm ext} = \frac{\partial^2_{\tilde{x}} h_{\rm ext}}{(1+(\partial_{\tilde{x}} h_{\rm ext})^2)^{3/2}}. \tag{7.69}$$

Using that the pressure is hydrostatic in the fluid substrate, we deduce from (7.68) that

$$\int_{\mathrm{QS}} \rho_{\mathrm{sub}} \mathrm{g} h_{\mathrm{ext}} \mathbf{n}_{\mathrm{ext}} \, d\tilde{s}_{\mathrm{ext}} + \gamma_{\mathrm{g/s}} (\boldsymbol{\tau}_{\mathrm{ext}}(\mathrm{Q}) - \boldsymbol{\tau}_{\mathrm{ext}}(\mathrm{S})) = 0.$$

Eliminating the contributions associated with the triple point, the resulting horizontal force on the control volume PQR is found to be

$$\int_{\mathrm{PR}} (\mathrm{p}\mathbf{I} - \eta \mathbf{d}) \cdot \mathbf{n}_{\mathrm{b}} \, dz - \mathbf{n}_{\mathrm{b}} \int_{\mathrm{QR}} \rho_{\mathrm{sub}} \mathrm{g} h_{\mathrm{sub}} \mathbf{n}_{\mathrm{sub}} \cdot \mathbf{n}_{\mathrm{b}} \, d\tilde{s}_{\mathrm{sub}}$$
$$- \mathbf{n}_{\mathrm{b}} \int_{\mathrm{QS}} \rho_{\mathrm{sub}} \mathrm{g} h_{\mathrm{ext}} \mathbf{n}_{\mathrm{ext}} \cdot \mathbf{n}_{\mathrm{b}} \, d\tilde{s}_{\mathrm{ext}} - \mathbf{n}_{\mathrm{b}} (\gamma_{\mathrm{f/s}} + \gamma_{\mathrm{f/g}} - \gamma_{\mathrm{g/s}}).$$

The horizontal projection of the surface tension force due to the substrate is easily evaluated as

$$\int_{\mathrm{QS}} \rho_{\mathrm{sub}} \mathrm{g} h_{\mathrm{ext}} \mathbf{n}_{\mathrm{ext}} \cdot \mathbf{n}_{\mathrm{b}} \, d\tilde{s}_{\mathrm{ext}} = \frac{1}{2} \rho_{\mathrm{sub}} \mathrm{g} (h_{\mathrm{ext}}^2(\mathrm{S}) - h_{\mathrm{ext}}^2(\mathrm{Q})), \tag{7.70}$$

since

$$\mathbf{n}_{\mathrm{ext}} \cdot \mathbf{n}_{\mathrm{b}} = \frac{\partial_{\tilde{x}} h_{\mathrm{ext}}}{\sqrt{1 + (\partial_{\tilde{x}} h_{\mathrm{ext}})^2}}, \qquad d\tilde{s}_{\mathrm{ext}} = \sqrt{1 + (\partial_{\tilde{x}} h_{\mathrm{ext}})^2} \, d\tilde{x},$$

and we may choose the vertical axis in such a way that $h_{\mathrm{ext}}(\mathrm{S}) = 0$ since pressures are measured relative to the atmospheric pressure. Upon defining

$$\gamma = \frac{\gamma_{\mathrm{g/s}} - \gamma_{\mathrm{f/s}} - \gamma_{\mathrm{f/g}}}{\rho} \tag{7.71}$$

and using the relations (7.65) and (7.70), the resulting horizontal force on the control volume PQR at zeroth-order reads

$$\int_{\mathrm{PR}} (\mathrm{p}_0 \mathbf{I} - \eta_0 \mathbf{d}_0) \cdot \mathbf{n}_{\mathrm{b}} \, dz - \mathbf{n}_{\mathrm{b}} \left(\frac{1}{2} \rho_{\mathrm{sub}} |\mathrm{g}_0| h_{\mathrm{sub}0}^2(\mathrm{R}) + \rho \gamma \right). \tag{7.72}$$

Since the control volume PQR is of the order of ϵ^2, we can neglect the inertial term and write that the resulting force (7.72) vanishes at zeroth-order.

The zeroth-order force $\int_{\mathrm{PR}} (\mathrm{p}_0 \mathbf{I} - \eta_0 \mathbf{d}_0) \cdot \mathbf{n}_{\mathrm{b}} \, dz$ can be evaluated from the expression of the zeroth-order strain tensor $\mathbf{d}_0$ and of the zeroth-order pressure p_0. Since the second-order tensor $\mathbf{d}_0$ restricted to the plane spanned by $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ can be written as $\boldsymbol{\partial_x} \boldsymbol{v}_0 + \boldsymbol{\partial_x} \boldsymbol{v}_0^t$ and is independent of z, we directly obtain upon integration that

$$\int_{\mathrm{PR}} \eta_0 \mathbf{d}_0 \cdot \mathbf{n}_{\mathrm{b}} \, dz = \eta_0 h_0 (\boldsymbol{\partial_x} \boldsymbol{v}_0 + \boldsymbol{\partial_x} \boldsymbol{v}_0^t) \cdot \mathbf{n}_{\mathrm{b}}.$$

On the other hand, from $\mathrm{p}_0 = -2\eta_0(\partial_x u_0 + \partial_y v_0) + \mathrm{g}_0 z - (1-\mathsf{a})\mathrm{g}_0 h_0$, we obtain upon integration that

$$-\int_{\mathrm{PR}} \mathrm{p}_0 \, dz = 2\eta_0 h_0(\partial_x u_0 + \partial_y v_0) + \frac{1}{2}\rho h_0^2 \mathrm{g}_0.$$

The last term $\frac{1}{2}\rho h_0^2 \mathrm{g}_0$ can then be combined with the contribution $\frac{1}{2}\rho_{\mathrm{sub}}|\mathrm{g}_0| h_{\mathrm{ext}}^2(\mathrm{R})$ from (7.72) to form the pressure term $\rho p_0 = \frac{1}{2}\rho(1-\mathsf{a})\mathrm{g}_0 h_0^2$ of the two-dimensional model. On the other hand, the term $2\eta_0 h_0(\partial_x u_0 + \partial_y v_0)$ will complete the isotropic part of $\boldsymbol{\Pi}_0$. Upon combining the above relations and dividing by the fluid density ρ we have finally established the dynamic boundary condition

$$-(p_0 \boldsymbol{I} + \boldsymbol{\Pi}_0) \cdot \mathbf{n}_{\mathrm{b}} = \gamma \mathbf{n}_{\mathrm{b}}, \tag{7.73}$$

where $\gamma = (\gamma_{\mathrm{g/s}} - \gamma_{\mathrm{f/s}} - \gamma_{\mathrm{f/g}})/\rho$.

Remark 7.3. It is interesting to note that in the zeroth-order governing equations the surface tensions do not appear. Surface tensions only play a role in the zeroth order dynamic boundary conditions.

We can proceed similarly for the thermal boundary condition by considering the control domain PQR. We observe then that the heat exchange coefficients are of order $\mathcal{O}(\epsilon)$ as are the length of the arcs PQ and QR in such a way that

$$\int_{\mathrm{PQ}} \mathbf{Q}_0 \cdot \mathbf{n}_{\mathrm{gas}} \, d\tilde{s}_{\mathrm{gas}} = \int_{\mathrm{PQ}} \lambda_{\mathrm{gas}}^*(T_0 - T_{\mathrm{gas}}) d\tilde{s}_{\mathrm{gas}} = \mathcal{O}(\epsilon^2),$$

$$\int_{\mathrm{QR}} \mathbf{Q}_0 \cdot \mathbf{n}_{\mathrm{sub}} \, d\tilde{s}_{\mathrm{sub}} = \int_{\mathrm{QR}} \lambda_{\mathrm{sub}}^*(T_0 - T_{\mathrm{sub}}) d\tilde{s}_{\mathrm{sub}} = \mathcal{O}(\epsilon^2),$$

where $\mathbf{Q}_0 = -\lambda_0 \boldsymbol{\partial}_{\mathbf{x}} T_0$ is the three-dimensional zeroth-order heat flux. Upon integrating the heat conservation equation in the domain PQR, we thus obtain at zeroth-order that

$$\int_{\mathrm{PR}} \lambda_0 \boldsymbol{\partial}_{\boldsymbol{x}} T_0 \cdot \mathbf{n}_{\mathrm{b}} \, dz = 0,$$

and therefore

$$\boldsymbol{Q}_0 \cdot \mathbf{n}_{\mathrm{b}} = 0, \tag{7.74}$$

where $\boldsymbol{Q}_0 = -h_0 \lambda_0 \boldsymbol{\partial}_{\boldsymbol{x}} T_0$ is the zeroth-order heat flux for the Saint-Venant model. Finally, since the free surface $x = X_{\mathrm{b}}(t, y)$ is a streamline of the two-dimensional flow model, we obtain the natural kinematic condition

$$u_0 = \partial_t X_{\mathrm{b}} + v_0 \partial_y X_{\mathrm{b}}.$$

This boundary condition can equivalently be obtained by integrating the incompressibility condition $\boldsymbol{\partial}_{\mathbf{x}} \cdot \mathbf{v} = 0$ on the control domain PQR and is easily rewritten in the coordinate independent form

$$h_0(\boldsymbol{v}_0 - \boldsymbol{v}_{\mathrm{b}}) \cdot \mathbf{n}_{\mathrm{b}} = 0, \tag{7.75}$$

where $\boldsymbol{v}_{\mathrm{b}} = \partial_t X_{\mathrm{b}}$ is the velocity of the free boundary and it is well known that the normal velocity $\boldsymbol{v}_{\mathrm{b}} \cdot \mathbf{n}_{\mathrm{b}}$ is an intrinsic quantity associated with the free boundary.

The boundary conditions for the two-dimensional Saint-Venant model at the free boundary are finally constituted by the kinematic condition (7.75), the dynamic momentum boundary condition (7.73), and the thermal condition (7.74).

Finally we would like to address the validity of this set of boundary conditions. From the above derivation, these conditions are physically relevant when the domain PQR is of size $\mathcal{O}(\epsilon^2)$, that is, when its horizontal diameter remains of size $\mathcal{O}(\epsilon)$. In this situation, there is an abrupt change at the free boundary of the three-dimensional domain and the resulting quantity h_0 can be considered to be positive up to the free boundary of the two-dimensional Saint-Venant system.

The physical origin of such a behavior is associated with wettability theory[15] and with the sign of the spreading parameter $\gamma_{\mathrm{g/s}} - \gamma_{\mathrm{f/s}} - \gamma_{\mathrm{f/g}}$. This parameter is *negative* for molten glass on molten tin or oil on water for instance. In this situation, from the dynamic boundary condition (7.73), we obtain that the equilibrium thickness $h_{0\mathrm{eq}}$ such that $p_0 = -\gamma$ is given by $h_{0\mathrm{eq}} = (2(\gamma_{\mathrm{f/s}} + \gamma_{\mathrm{f/g}} - \gamma_{\mathrm{g/s}})/(1 - \mathsf{a})\mathrm{g}_0\rho)^{1/2}$ as discussed by de Genes, Brochard-Wyart, and Quéré.[15] Surface tension therefore prevents h_0 to be too small when the spreading parameter is negative and this is the case for float glasses where the equilibrium thickness is $h_{0\mathrm{eq}} \approx 7\,\mathrm{mm}$ or octane on water where the equilibrium thickness is $h_{0\mathrm{eq}} \approx 3.7\,\mathrm{mm}$.[15] On the contrary, the spreading parameter is positive for Polydimethylsiloxanes on water.[15]

Finally, from a mathematical point of view, if we neglect the surface tension, we note that these boundary conditions disappear when $h_0 = 0$, that is, in the presence of a vacuum state, as they should as established by Sanchez-Hubert and Sanchez Palencia,[51] Bresch,[8] and Li *et al.*[41] in various frameworks in the isothermal case.

References

1. E. Audusse, A multilayer Saint-Venant model, *Disc. Cont. Dynam. Syst.* **5** (2005) 189–214.
2. E. Audusse and M. O. Bristeau, Finite volume solvers for a multilayer Saint-Venant system, *Int. J. Appl. Math. Comput. Sci.* **17** (2007) 311–320.
3. J. P. Benqué, A. Haugel and P. L. Viollet, Engineering applications of computational hydraulics, in Pitman Advanced Publishing Program, Vol. II (Pitman, 1982), pp. 57–63.
4. A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis of Periodic Structure* (North-Holland, 1978).
5. C. Bernardi and O. Pironneau, On the shallow water equations at low Reynolds number, *Comm. Part. Diff. Eqns.* **16** (1991) 59–104.
6. M. S. Baouendi and C. Goulaouic, Régularité et théorie spectrale pour une classe d'opérateurs elliptiques dégénérés, *Arch. Rational Mech. Anal.* **34** (1969) 361–379.
7. G. Billet, V. Giovangigli and G. de Gassowski, Impact of volume viscosity on a shock/hydrogen bubble interaction, *Comb. Th. Model* **12** (2008) 221–248.
8. D. Bresch, Shallow water equations and related topics, in *Evolutionary Equations*, eds. C. M. Dafermos and M. Pokorný, Handbook of Differential Equations, Vol. 5 (Elsevier, 2009).

9. D. Bresch and B. Desjardins, Existence of global weak solutions for a 2D viscous shallow water equations and convergence to the quasi-geostropic model, *Comm. Math. Phys.* **238** (2003) 211–223.
10. D. Bresch, B. Desjardins and G. Métivier, Recent mathematical results and open problems about shallow water equations, in *Analysis and Simulation of Fluid Dynamics*, Series in Advances in Mathematical Fluid Mechanics (Birkhäuser, 2006), pp. 15–31.
11. D. Bresch and P. Noble, Mathematical justification of a shallow water model, *Meth. Appl. Anal.* **14** (2007) 87–118.
12. G. F. Carey, *Finite Element Modeling of Environmental Problems, Surface and Subsurface Flow and Transport* (John Wiley and Sons, 1995).
13. F. Chalot, T. J. R. Hughes and F. Shakib, Symmetrization of conservation laws with entropy for high-temperature hypersonic computations, *Comp. Syst. Engrg.* **1** (1990) 495–521.
14. C. Dafermos, *Hyperbolic Systems of Conservation Laws* (Springer-Verlag, 1997).
15. P. G. de Genes, F. Brochard-Wyart and D. Quéré, *Capillarity and Wetting Phenomena* (Springer-Verlag, 2004).
16. A. J. C. de Saint-Venant, Théorie du mouvement non permanent des eaux, avec application aux crues des rivìres et à l'introduction des marées dans leur lit, *C. R. Acad. Sci. Paris* **73** (1871) 147–154.
17. K. O. Friedrichs and P. D. Lax, Systems of conservation laws with a convex extension, *Proc. Natl. Acad. Sci. USA* **68** (1971) 1686–1688.
18. P. R. Gent, The energetically consistent shallow water equations, *J. Atm. Sci.* **50** (1993) 1323–1325.
19. J. F. Gerbeau and B. Perthame, Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation, *Disc. Cont. Dynam. Syst.* **1** (2001) 89–102.
20. V. Giovangigli, *Multicomponent Flow Modeling* (Birkhäuser, 1999) [Erratum at the internet site http://cmap.polytechnique.fr/∼giovangi].
21. V. Giovangigli, Higher order entropies, *Arch. Rational Mech. Anal.* **187** (2008) 221–285.
22. V. Giovangigli, Higher order entropies for compressible fluid models, *Math. Models Methods Appl. Sci.* **19** (2009) 67–125.
23. V. Giovangigli and B. Graille, Asymptotic stability of equilibrium states for ambipolar plasmas, *Math. Models Methods Appl. Sci.* **14** (2004) 1361–1399.
24. V. Giovangigli and B. Graille, The local Cauchy problem for ionized magnetized reactive gas mixtures, *Math. Methods Appl. Sci.* **28** (2005) 1647–1672.
25. V. Giovangigli and M. Massot, Asymptotic stability of equilibrium states for multicomponent reactive flows, *Math. Models Methods Appl. Sci.* **8** (1998) 251–297.
26. V. Giovangigli and M. Massot, The local Cauchy problem for multicomponent reactive flows in full vibrational nonequilibrium, *Math. Methods Appl. Sci.* **21** (1998) 1415–1439.
27. V. Giovangigli and M. Massot, Entropic structure of multicomponent reactive flows with partial equilibrium reduced chemistry, *Math. Methods Appl. Sci.* **27** (2004) 739–768.
28. S. K. Godunov, The problem of a generalized solution in the theory of quasilinear equations and in gas dynamics, *Russ. Math. Surv.* **17** (1962) 145–156.
29. G. Hauke, A symmetric formulation for computing transient shallow water flows, *Comp. Meth. Appl. Mech. Engrg.* **163** (1998) 111–122.
30. D. P. Hoult, Oil spreading on the sea, *Ann. Rev. Fluid Mech.* (1972) 341–367.
31. P. D. Howell, Extensional thin layer flows, Ph.D. Thesis, St. Catherine College, Cambridge University (1994).
32. P. D. Howell, Models for thin viscous sheet, *Euro. J. Appl. Math.* **7** (1996) 1–23.
33. T. J. R. Hughes, L. P. Franca and M. Mallet, A new finite element formulation for computational fluid dynamics: I. Symmetric forms of the compressible Euler and

Navier–Stokes equations and the second law of thermodynamics, *Comp. Meth. Appl. Mech. Engrg.* **54** (1986) 223–234.
34. H. Kanayama and T. Ushijima, *On the Viscous Shallow Water Equations I, Derivation and Conservation Laws*, Memoirs of Numerical Math., Vols. 8/9 (1981/1982), pp. 39–64.
35. H. Kanayama and T. Ushijima, *On the Viscous Shallow Water Equations II, A Linearized System*, Bull. Univ. Electro-Comm., Vol. 1 (1988), pp. 347–355.
36. H. Kanayama and H. Dan, A finite element scheme for two-layer viscous shallow-water equations, *Jpn. J. Indust. Appl. Math.* **23** (2006) 163–191.
37. S. Kawashima, Systems of hyperbolic–parabolic composite type, with application to the equations of magnetohydrodynamics, Doctoral Thesis, Kyoto University (1984).
38. S. Kawashima and Y. Shizuta, On the normal form of the symmetric hyperbolic–parabolic systems associated with the conservation laws, *Tôhoku Math. J.* **40** (1988) 449–464.
39. P. E. Kloeden, Global existence of classical solutions in the dissipative shallow water equations, *SIAM J. Math. Anal.* **16** (1985) 301–315.
40. D. Levermore and B. Sammartino, A shallow water model in a basin with varying bottom topography and addy viscosity, *Nonlinearity* **14** (2001) 1493–1515.
41. H. L. Li, J. Li and Z. Xin, Vanishing of vacuum states and blow-up phenomena of the compressible Navier–Stokes equations, *Comm. Math. Phys.* **281** (2008) 401–444.
42. P. L. Lions, *Mathematical Topics in Fluid Mechanics* Vols. 1 and 2, Oxford Lecture Series in Mathematics and its Applications (Oxford Univ. Press, 1996 and 1998).
43. M. L. Muñoz-Ruiz, On a non-homogeneous bi-layer shallow-water problem: Smoothness and uniqueness results, *Nonlinear Anal.* **59** (2004) 253–282.
44. M. Mock, Systems of conservation laws of mixed type, *J. Diff. Eqns.* **37** (1980) 70–88.
45. P. Orenga, Un théorème d'existence de solutions d'un problème de shallow water, *Arch. Rational Mech. Anal.* **130** (1995) 183–204.
46. E. E. Paladino and C. R. Maliska, Mathematical modeling and numerical simulation of oil spill trajectories on the sea, in CONEM 2000 (2000).
47. L. A. B. Pilkington, The float glass process, *Proc. Roy. Soc. Lond.* **314** (1969) 1–25.
48. V. Podsetchine, G. Schernewski and I. Tejakusuma, Simulations of water flows, transport, and temperature, in the Oder (Szczecin) Lagoon, Marine Science Reports 57 (2004).
49. M. Roseau, *Vibrations in Mechanical Systems* (Springer-Verlag, 1987).
50. E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Lecture Notes in Physics, Vol. 127 (Springer-Verlag, 1980).
51. J. Sanchez-Hubert and E. Sanchez-Palencia, *Vibration and Coupling of Continuous Systems* (Springer-Verlag, 1989).
52. D. Serre, Systèmes de lois de conservation I and II, Diderot, Ed., Arts et Sciences, Paris (1996).
53. Y. Shizuta and S. Kawashima, Systems of equations of hyperbolic–parabolic type with applications to the discrete Boltzmann equation, *Hokkaido Math. J.* **14** (1985) 249–275.
54. L. Sundbye, Global existence for the Dirichlet problem for the viscous shallow water equations, *J. Math. Anal. Appl.* **202** (1996) 236–258.
55. E. Tadmor, Skew-self-adjoint form for systems of conservation laws, *J. Math. Anal. Appl.* **103** (1984) 428–442.
56. B. A. Ton, Existence and uniqueness of a classical solution of an initial boundary value problem of the theory of shallow waters, *SIAM J. Math. Anal.* **12** (1981) 229–241.
57. M. Van Dyke, *Perturbation Methods in Fluid Mechanics* (Academic Press, 1964).
58. A. I. Vol'pert and S. I. Hudjaev, On the Cauchy problem for composite systems of non-linear differential equations, *Math. USSR Sbornik* **16** (1972) 517–544.
59. W. Wang and C. J. Xu, The Cauchy problem for viscous shallow water flows, *Rev. Mat. Iber.* **21** (2005) 1–24.