Functional analysis and applications
MASTER "Mathematical Modelling"
École Polytechnique and Université Pierre et Marie Curie
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See also the course webpage:
http://www.cmap.polytechnique.fr/ allaire/master/course-funct-analysis.html

Exercise 1

Let E be a Banach space and let (x_n) be a sequence such that $x_n \rightharpoonup x$ in the weak $\sigma(E, E^*)$ topology. Set

$$y_n = \frac{1}{n} \sum_{k \leq n} x_k.$$

Prove that $y_n \rightharpoonup x$.

Answer of exercise 1

Let $T \in E^*$. We have

$$T(y_n) = \frac{1}{n} \sum_{k \leq n} T(x_k).$$

and

$$|T(y_n) - T(x)| \leq \frac{1}{n} \sum_{k \leq n} |T(x_n) - T(x)|.$$

Fro all $\varepsilon > 0$, there exists N such that for all $n > N$, $|T(x_n) - T(x)| < \varepsilon$. Thus,

$$|T(y_n) - T(x)| \leq \frac{1}{n} \sum_{k \leq N} |T(x_n) - T(x)| + \varepsilon,$$

and for n great enough,

$$|T(y_n) - T(x)| \leq 2\varepsilon.$$

Exercise 2

Let $\Omega = (0, 1)$.

1. Consider the sequence (f_n) of functions defined by $f_n(x) = ne^{-nx}$. Prove that
   1. $f_n \to 0$ a.e.
   2. (f_n) is bounded in $L^1(\Omega)$.
   3. $f_n \not\to 0$ in $L^1(\Omega)$ strongly.
   4. $f_n \not\rightharpoonup 0$ weakly in $\sigma(L^1, L^\infty)$.

   More precisely, there is no subsequence that converges weakly $\sigma(L^1, L^\infty)$.

2. Let $1 < p < \infty$ and consider the sequence (g_n) of functions defined by $g_n(x) = n^{1/p}e^{-nx}$. Prove that

   1. $g_n \to 0$ a.e.
   2. (g_n) is bounded in $L^p(\Omega)$.
   3. $g_n \not\to 0$ in $L^p(\Omega)$ strongly but $g_n \to 0$ in $L^q(\Omega)$ strongly for every $1 \le q < p$.
   4. $g_n \rightharpoonup 0$ weakly in $\sigma(L^p, L^{p'})$.

Answer of exercise 2

1. We consider $f_n = ne^{-nx}$. For all $x \in (0,1)$, $f_n(x) = e^{-n(x+\ln(n)/n)}$ and converges toward 0.

$$\|f_n\|_{L^1(\Omega)=} \int_0^1 |f_n| = \int_0^1 ne^{-nx} = -\int_0^1 (e^{-nx})' = -[e^{-nx}]_0^1 = 1 - e^{-n}.$$

Thus, f_n is bounded in $L^1(\Omega)$ and does not converge toward 0 in $L^1(\Omega)$. Finally, let $u \in C^1([0,1])$,

$$\int_0^1 f_n u = \int_0^1 (e^{-nx})'u = [e^{-nx}u]_0^1 - \int_0^1 e^{-nx}u' \to u(0).$$

Thus, f_n does not converge toward 0 in $\sigma(L^1, L^\infty)$ (and even no subsequence).

2. We have defined g_n by

$$g_n(x) = n^{1/p}e^{-nx},$$

with $1 < p < \infty$. Obviously, $g_n(x)$ goes to zero for every $x \in (0,1)$. Moreover,

$$\|g_n\|_{L^p}^p = \int_0^1 ne^{-npx} = \frac{1}{p}(1 - e^{-np}).$$

Thus, g_n is bounded in $L^p(0,1)$ and does not converge toward 0 in $L^p(0,1)$. Now, let $u \in C_0^\infty(0,1)$. As g_n does converge uniformly toward 0 on the support of u, we have

$$\int_0^1 g_n u \to 0.$$

Let $v \in L^{p'}(0,1)$. For all $\varepsilon > 0$. there exists $u_\varepsilon \in C_0^\infty(0,1)$ such that $\|u_\varepsilon - v\|_{L^{p'}(\Omega)} < \varepsilon$. It follows that

$$\left|\int_0^1 g_n v\right| \le \left|\int_0^1 g_n(u_\varepsilon - v)\right| + \left|\int_0^1 g_n u_\varepsilon\right| \le \|g_n\|_{L^p}\|u_n - v\|_{L^{p'}} + \left|\int_0^1 g_n u_\varepsilon\right|.$$

As g_n is bounded in $L^p(0,1)$, we get that

$$\left|\int_0^1 g_n v\right| \le C\varepsilon + \left|\int_0^1 g_n u_\varepsilon\right|.$$

For n great enough, we obtain that

$$\left|\int_0^1 g_n v\right| \leq (C+1)\varepsilon.$$

It follows that $\int_0^1 g_n v$ converges toward 0 as n goes to infinity, that is g_n converges weak toward 0 in $L^p(0,1)$.

Exercise 3

Assume that $|\Omega| < \infty$. Let $1 < p < \infty$. Let (f_n) be a sequence in $L^p(\Omega)$ such that

1. (f_n) is bounded in $L^p(\Omega)$.
2. $f_n \to f$ a.e. on Ω.

1. Prove that $f_n \rightharpoonup f$ weakly in $\sigma(L^p, L^{p'})$.
2. Same conclusion if assumption (2) is replaced by

$$\|f_n - f\|_1 \to 0.$$

3. Assume now (1) and (2) and $|\Omega| < \infty$. Prove that $\|f_n - f\|_q \to 0$ for every q with $1 \leq q < p$.

Answer of exercise 3

1. To simplify the proof, we assume that $|\Omega| < \infty$. It can be easily adapt to the case where Ω is σ-finite. First, let us notice that that from Fatou's Lemma, $f \in L^p(\Omega)$. In a first step, we are going to prove that up to a subsequence, f_n weakly converges toward f in $L^p(\Omega)$. As f_n is bounded in $L^p(\Omega)$, it admits a weakly convergent subsequence. That is there exists φ monotone map from $\mathbb{N}$ into $\mathbb{N}$ and $\widetilde{f} \in L^p(\Omega)$ such that $f_{\varphi(n)}$ weakly converges toward $\widetilde{f}$. Moreover, from the Egorov's Theorem, for all integer $m > 0$, there exists a measurable subset A_m of Ω such that $f_{\varphi(n)}$ converges toward f uniformly. It follows that for all $g \in L^{p'}(\Omega)$,

$$\int_{\Omega\setminus A_m} f_{\varphi(n)} g \to \int_{\Omega\setminus A_m} fg$$

and

$$\int_{\Omega\setminus A_m} f_{\varphi(n)} g \to \int_{\Omega\setminus A_m} \widetilde{f} g.$$

Thus,

$$\int_{\Omega\setminus A_m} (f - \widetilde{f}) g = 0,$$

for every $g \in L^{p'}(\Omega)$. Choosing $g = \text{sign}(f - \widetilde{f})$, it follows that $f = \widetilde{f}$ a.e. in $\Omega \setminus A_m$. In particular, $f = \widetilde{f}$ a.e. in $\Omega \setminus (\cap_m A_m)$. As $|\cap_m A_m| = 0$, we deduce that $f = \widetilde{f}$ almost everywhere. It remains to prove that the whole sequence f_n weakly converges toward f in $L^p(\Omega)$. Assume this is not the case. Then, there exists $h \in L^{p'}(\Omega)$ and $\psi : \mathbb{N} \to \mathbb{N}$ monotone such that

$$\left| \int_\Omega (f_{\psi(n)} - f)h \right| > \delta > 0.$$

Replacing f_n by $f_{\psi(n)}$ in the first part of the proof, we conclude that there exists $\varphi ; \mathbb{N} \to \mathbb{N}$ monotone such that

$$f_{\varphi \circ \psi(n)} \rightharpoonup f \text{ in } L^p(\Omega)$$

and

$$\left| \int_\Omega (f_{\varphi \circ \psi(n)} - f)h \right| > \delta > 0,$$

what is contradictory. We conclude as the whole sequence f_n weakly converges toward f in $L^p(\Omega)$.

2. The proof is exactly the same as in the previous case. It departs only in to establish that $\widetilde{f} = f$. In this case, we have immediately that

$$\int_\Omega (f - \widetilde{f})g = 0,$$

for all $g \in L^\infty(\Omega)$. Choosing once again $g = \text{sign}(f - \widetilde{f})$, we get that $f = \widetilde{f}$ a.e.

3. For every $\varepsilon > 0$, from the Egorov's Theorem, there exists a measurable subset A of Ω, such that $|A| < \varepsilon$ and f_n converges uniformly toward f in $\Omega \setminus A$. We have

$$\int_\Omega |f_n - f|^q = \int_A |f_n - f|^q + \int_{\Omega \setminus A} |f_n - f|^q.$$

From Hölder's inequality, we have

$$\int_A |f_n - f|^q \leq \left(\int_A |f_n - f|^p \right)^{q/p} |A|^{\frac{p-q}{p}} < C\varepsilon^{\frac{p-q}{p}}.$$

Moreover, as f_n uniformly converges toward f on $\Omega \setminus A$, for n great enough, we have

$$\int_{\Omega \setminus A} |f_n - f|^q < \varepsilon.$$

We conclude that for n great enough,

$$\int_\Omega |f_n - f|^q < C\varepsilon^{\frac{p-q}{p}} + \varepsilon,$$

and that f_n converges toward f in $L^q(\Omega)$ for all $1 \leq q < p$.

Exercise 4

Let E be a Banach space and $x_0 \in E$ be fixed. Prove that there exists $T \in E^*$ such that

$$T(x_0) = \|x_0\|_E^2$$

and $\|T\|_{E^*} = \|x_0\|_E$.

Answer of exercise 4

Let $G = \mathbb{R}x$ and T be the linear continuous map defined on G by

$$T(tx_0) = t\|x_0\|^2.$$

From the Hahn-Banach Theorem, there exists an extension of T on E such that $\|T\|_{E^*} = \|T\|_{G^*} = \|x_0\|_E$.

Exercise 5

Let E be a Banach space and let $A \subset E$ be a subset that is sequentially compact for the weak topology of E. Prove that A is bounded.

Answer of exercise 5

Let (x_n) be a sequence of elements of A. As A is sequentially compact, there exists $\varphi : \mathbb{N} \to \mathbb{N}$ such that φ is increasing and $x \in A$, with

$$x_{\varphi(n)} \rightharpoonup x.$$

In particular, for all $T \in E^*$, $T(x_{\varphi(n)})$ is bounded and, from the Banach-Steinhauss Theorem, there exists C such that for all $T \in E^*$,

$$T(x_{\varphi(n)}) \leq C\|T\|_{E^*}.$$

From the Exercise 4, there exists T such $T(x_{\varphi(n)}) = \|x_{\varphi(n)}\|^2$ and $\|T\| = \|x_{\varphi(n)}\|$. It follows that

$$\|x_{\varphi(n)}\| \leq C.$$

If A was not bounded, we could construct a sequence x_n of elements of A such that $\|x_n\|_E \geq n$, what is impossible from the last inequality.

Exercise 6 Rademacher's functions

Let $1 \leq p \leq \infty$ and let $f \in L^p_{\mathrm{loc}}(\mathbb{R})$. Assume that f is T-periodic, i.e., $f(x+T) = f(x)$, a.e. on $\mathbb{R}$. Set

$$\overline{f} = |T|^{-1} \int_0^T f(t)dt.$$

Consider the sequence (u_n) in $L^p(0,1)$ defined by

$$u_n(x) = f(nx), \qquad x \in (0,1).$$

1. Prove that $u_n \rightharpoonup \overline{f}$ with respect to the topology $\sigma(L^p, L^{p'})$.
2. Determine $\lim_{n\to\infty} \|u_n - \overline{f}\|_p$.
3. Examine the following examples:
    1. $u_n(x) = \sin(nx)$.
    2. $u_n(x) = f_n(x)$ where f is 1-periodic and

$$f(x) = \begin{cases} \alpha & \text{for } x \in (0, 1/2), \\ \beta & \text{for } x \in (1/2, 1). \end{cases}$$

The functions of (2) are called *Rademacher's functions.*

Answer of exercise 6

1. Let $0 < a < b < 1$ and v be the indicator function of (a, b) on $(0, 1)$, that is

$$v(x) = \begin{cases} 1 & \text{if } a < x < b, \\ 0 & \text{if } x \in (0,1) \setminus (a, b). \end{cases}$$

We have

$$\int_0^1 u_n v = \int_0^1 f(nx)v(x)\,dx = n^{-1}\int_0^n f(x)v(x/n)\,dx = n^{-1}\int_{na}^{nb} f(x)\,dx.$$

We set k and l to be the integers such that

$$(k-1)T < na \le kT, \qquad lT < nb \le (l+1)T.$$

We have

$$\begin{aligned} \int_0^1 u_n v &= \frac{1}{n}\left[\int_{na}^{kT} f + \sum_{k\le i\le l-1}\int_{iT}^{(i+1)T} f + \int_{lT}^{nb} f\right] \\ &= \frac{1}{n}\left[\sum_{k\le i\le l-1}\int_0^T f\right] + \frac{1}{n}\left[\int_{na}^{kT} f + \int_{lT}^{nb} f\right] \\ &= \frac{l-k}{n}\int_0^T f + \frac{1}{n}\left[\int_{na}^{kT} f + \int_{lT}^{nb} f\right]. \end{aligned}$$

From the definition of k and l, we have

$$\frac{l-k}{n} \le \frac{b-a}{T} \le \frac{l-k+2}{n}.$$

Thus, $(l-k)/n \to (b-a)/T$. Moreover,

$$\frac{1}{n}\left|\int_{na}^{kT} f + \int_{lT}^{nb} f\right| \le \frac{2}{n}\|f\|_{L^1(0,T)} \to 0.$$

It follows that
$$\int_0^1 u_n v \to \overline{f}(b-a),$$
as n goes to infinity. We deduce that for any step function v, we have
$$\int_0^1 u_n v \to \overline{f}\int_0^1 v.$$
As the set of step functions is dense in $L^{p'}(0,1)$, with $1 \leq p' < \infty$, we deduce that if $f \in L^p_{loc}(0,T)$, with $1 < p \leq \infty$, u_n does converge toward $\overline{f}$ in $\sigma(L^p, L^{p'})$. Indeed, for every $v \in L^{p'}(0,1)$ and for every $\varepsilon > 0$, there exists a step function w such that $\|v-w\|_{L^{p'}(0,1)} \leq \varepsilon$. We then have
$$\left|\int_0^1 u_n v - \overline{f}\int_0^1 v\right| \leq \left|\int_0^1 u_n w - \overline{f}\int_0^1 w\right| + \int_0^1 |u_n||v-w| + |\overline{f}|\int_0^1 |v-w|.$$
From Hölder inequality,
$$\int_0^1 |u_n||v-w| \leq \|u_n\|_{L^p(0,1)}\|v-w\|_{L^{p'}(0,1)}$$
and
$$|\overline{f}|\int_0^1 |v-w| \leq |\overline{f}|\|v-w\|_{L^{p'}(0,1)}.$$
Moreover, from the previous analysis, we have
$$\|u_n\|^p_{L^p(0,1)} = \int_0^1 u_n^p\,dx \to T^{-1}\int_0^T f^p.$$
In particular, u_n is bounded in $L^p(0,1)$. We have obtained that
$$\left|\int_0^1 u_n v - \overline{f}\int_0^1 v\right| \leq \left|\int_0^1 u_n w - \overline{f}\int_0^1 w\right| + C\|v-w\|_{L^{p'}(0,1)}.$$
Finally, has w is a step function, for n great enough,
$$\left|\int_0^1 u_n w - \overline{f}\int_0^1 w\right| \leq \varepsilon$$
and
$$\left|\int_0^1 u_n v - \overline{f}\int_0^1 v\right| \leq (1+C)\varepsilon.$$
It remains to consider the case $p = 1$ and $f \in L^1_{loc}(\mathbb{R})$. For every $\varepsilon > 0$, there exists a T periodic function, $g \in L^\infty$ such that
$$T^{-1}\|f-g\|_{L^1(0,T)} \leq \varepsilon$$

For every $v \in L^\infty(0,1)$, we have from the previous analysis,

$$\int_0^1 g(nx)v(x)\,dx \to \overline{g}\int_0^1 v.$$

On the hand, we have

$$\left|\int_0^1 u_n v - \overline{f}\int_0^1 v\right| \leq \|f(nx)-g(nx)\|_{L^1(0,1)}\|v\|_\infty + \left|\int_0^1 g(nx)v - \overline{f}\int_0^1 v\right|$$

We have

$$\|f(nx)-g(nx)\|_{L^1(0,1)} \to \frac{1}{T}\int_0^T |f-g| \leq \varepsilon.$$

Thus, for n great enough, we have

$$\|f(nx) - g(n(x)\|_{L^1(0,1)} \leq 2\varepsilon$$

and

$$\left|\int_0^1 g(nx)v - \overline{f}\int_0^1 v\right| \leq \varepsilon + |\overline{f} - \overline{g}|\|v\|_{L^1(0,1)}.$$

Furthermore

$$|\overline{f} - \overline{g}| \leq T^{-1}\int_0^T |f-g| \leq \varepsilon.$$

We thus have proved that for n great enough

$$\left|\int_0^1 u_n v - \overline{f}\int_0^1 v\right| \leq 2\varepsilon\|v\|_\infty + \varepsilon + \varepsilon\|v\|_1,$$

and that $\int u_n v \to \overline{f}\int v$ as claimed.

2. We set $g(s) = |f(s) - \overline{f}|^p$. As g is T-periodic ans $g \in L^1_{loc}(\mathbb{R})$, we have from the previous question

$$\int_0^1 g(nx)\,dx \to \overline{g},$$

that is

$$\lim \|u_n - \overline{f}\|^p = \frac{1}{|T|^2}\int_0^T \left|\int_0^T (f(s)-f(t))\,ds\right|^p dt.$$

3. 1. $u_n = \sin(nx)$. We have $u_n \rightharpoonup 0$ weakly-* in L^∞,
   2. $u_n = f(nx)$ where f is one periodic and

$$f(x) = \begin{cases} \alpha & \text{if } x \in (0,1/2) \\ \beta & \text{if } x \in (1/2,1). \end{cases}$$

   Then $u_n \rightharpoonup (\alpha+\beta)/2$ for the weak-* topology of $L^\infty(0,1)$.