Ecole Polytechnique, Promotion 2008
Numerical analysis and optimization (MAP 431)
April 13th, 2010, G. Allaire

1 Finite differences (7 points)

We consider the heat equation with periodic boundary conditions in $(0,1)$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = 0 \text{ for } (x,t) \in (0,1) \times \mathbb{R}^+_*, \\ u(t, x+1) = u(t,x) \text{ for } (x,t) \in \mathbb{R} \times \mathbb{R}^+_*, \\ u(0,x) = u_0(x) \text{ for } x \in (0,1), \end{cases} \tag{1}$$

with $\nu > 0$. Let $\Delta t > 0$ and $\Delta x = 1/N > 0$ (with a positive integer N) and define the nodes of a regular mesh

$$(t_n, x_j) = (n\Delta t, j\Delta x) \text{ for } n \geq 0, j \in \mathbb{Z}.$$

We denote by u_j^n a discrete approximation at the point (t_n, x_j) of the exact solution $u(t,x)$. We consider the following scheme

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \nu \frac{-u_{j+1}^n + 2u_j^{n+1} - u_{j-1}^n}{(\Delta x)^2} = 0,$$

with an initial data $u_j^0 = u_0(x_j)$ and a boundary condition $u_{j+N}^n = u_j^n$, $\forall j$.

1. Analyze the stability of this numerical scheme.
2. Show that the scheme is consistent under a CFL-type condition which has to be made precise.
3. What can be said on the convergence of this scheme ? Discuss its pro's and con's compared to other classical schemes studied in the course.

2 Variational formulation (13 points)

We consider two materials occupying a domain $\Omega \subset \mathbb{R}^N$ (a bounded smooth connected open set), separated by an imperfect interface. The first material, with thermal conductivity $k_1 > 0$, occupies the connected complement $\Omega_1 = \Omega \setminus \overline{\Omega_2}$ of a smooth simply-connected subset Ω_2, strictly included in Ω, which contains the second material, with thermal conductivity $k_2 > 0$. The two sub-domains are separated by an interface $\Gamma = \partial\Omega_2$ which is a smooth surface. We thus have $\Omega = \Omega_1 \cup \Gamma \cup \Omega_2$ and the boundary of Ω_1 is defined by $\partial\Omega_1 = \partial\Omega \cup \Gamma$ (see figure 1). We denote by n_1 (respectively n_2) the exterior unit normal to Ω_1 (resp. Ω_2), f_1 (resp. f_2) the heat source term in Ω_1 (resp. Ω_2) and u_1 (resp. u_2) the temperature in Ω_1 (resp. Ω_2). The exterior of Ω is assumed to be kept at a constant temperature which, without loss of generality, is chosen to be zero ; in other words we consider an homogeneous Dirichlet boundary condition on $\partial\Omega$.

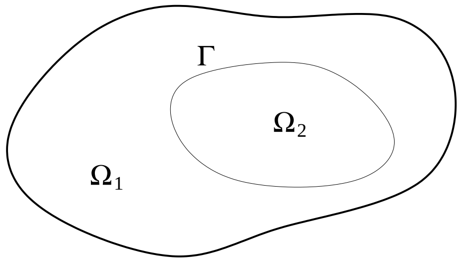

FIG. 1 – Two materials separated by the interface Γ.

The imperfect character of Γ means that the temperature **is not continuous** through Γ. The energy conservation implies that the heat flux is continuous through Γ and it is assumed to be proportional to the temperature jump through Γ with a proportionnality factor $\alpha > 0$. In othe words, we study the following coupled system

$$\begin{cases} -k_1 \Delta u_1 = f_1 & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \partial\Omega, \\ -k_1 \dfrac{\partial u_1}{\partial n_1} = \alpha(u_1 - u_2) & \text{on } \Gamma, \end{cases} \tag{2}$$

and

$$\begin{cases} -k_2 \Delta u_2 = f_2 & \text{in } \Omega_2, \\ -k_2 \dfrac{\partial u_2}{\partial n_2} = \alpha(u_2 - u_1) & \text{on } \Gamma. \end{cases} \tag{3}$$

We assume that $f_1(x)$ (resp. $f_2(x)$) belongs to $L^2(\Omega_1)$ (resp. $L^2(\Omega_2)$).

1. In this question (only) we assume that the temperature $u_2 \in L^2(\Gamma)$ is known. Give the variational formulation of (2). Prove the existence and uniqueness of the solution u_1 of this variational formulation. Assuming that this solution u_1 belongs to $H^2(\Omega_1)$, in which sense is it a solution of (2) too ?
2. Prove by contradiction that there exists a constant $C > 0$ such that
$$\forall v \in H^1(\Omega_2) \qquad \|v\|_{L^2(\Omega_2)} \leq C \left(\|\nabla v\|_{L^2(\Omega_2)^N} + \|v\|_{L^2(\Gamma)} \right).$$
3. In this question (only) we assume that the temperature $u_1 \in L^2(\Gamma)$ is known. Give the variational formulation of (3). Deduce from the preceding question the existence and uniqueness of the solution u_2 of this variational formulation.
4. Give the variational formulation of the coupled system (2)-(3). Prove the existence and uniqueness of the solution (u_1, u_2) of this variational formulation. Hint : one could use (after proving it) the following inequality, valid for any $\epsilon > 0$, as small as we wish,
$$(a_1 - a_2)^2 \geq -\epsilon a_1^2 + \frac{\epsilon}{1+\epsilon} a_2^2 \quad \forall a_1, a_2 \in \mathbb{R}.$$
5. What happens formally when α goes to $+\infty$? And when $\alpha = 0$?