ECOLE POLYTECHNIQUE – Promotion 2008
Analyse Numérique et Optimisation (MAP431)
Contrôle classant
Lundi 28 juin 2010

Durée : 4 heures

Sujet Proposé par François Alouges et Habib Ammari

The subject is composed of two problems that are independent one from another. It is 8 pages long. Each problem has to be solved on paper sheets of different colors, pink for problem 1 and green for problem 2.

Problem 1 - (Pink paper, 12 points)

Notations and recalls.
We consider Ω a bounded regular open set (of class $\mathcal{C}^\infty$) of $\mathbb{R}^d$, and $L^2(\Omega)$ the space of square integrable functions on Ω. We denote by

$$< u, v >= \int_\Omega u(x)v(x)\,dx, \text{ and } ||u||_{L^2} = (< u, u >)^{\frac{1}{2}}$$

respectively the scalar product of two functions u and v of $L^2(\Omega)$ and the associated norm. We also denote by $H_0^1(\Omega)$ the clasical Sobolev space of functions admitting a (weak) derivative in $L^2(\Omega)$ and whose trace vanishes on the boundary $\partial\Omega$. We equip $H_0^1(\Omega)$ with the scalar product

$$\forall u, v \in H_0^1(\Omega),\ a(u, v) = \int_\Omega \nabla u(x) \cdot \nabla v(x)\,dx\,,$$

and we denote by $||u||_{H_0^1}$ the associated norm. In the sequel, we will set $\lambda(u) = ||u||_{H_0^1}^2 = \displaystyle\int_\Omega |\nabla u(x)|^2\,dx$ for $u \in H_0^1(\Omega)$.

We recall that there exists a sequence of functions $(\phi_k)_{k\in\mathbb{N}}$ in $H_0^1(\Omega)$ which verify

$$\left[\begin{array}{l} -\Delta\phi_k = \lambda_k\phi_k \text{ in } \Omega \\ \phi_k = 0 \text{ on } \partial\Omega \\ ||\phi_k||_{L^2(\Omega)} = 1 \end{array}\right. \tag{1}$$

for an increasing sequence of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots$ which tends to $+\infty$. The functions ϕ_k are called the eigenfunctions of the Laplacian. Moreover, $(\phi_k)_{k\in\mathbb{N}}$ (respectively $\left(\frac{\phi_k}{\sqrt{\lambda_k}}\right)_{k\in\mathbb{N}}$) is a hilbert basis of $L^2(\Omega)$ (respectively of $H_0^1(\Omega)$).

For all $N \in \mathbb{N}^*$ we will call

$$E_N = \text{span}\{\phi_k,\ 1 \leq k \leq N\} \tag{2}$$

the $N-$dimensional subspace of $H_0^1(\Omega)$ generated by the first N eigenfunctions of the Laplacian.

In all the problem, we assume that λ_1 is simple, that is to say that

$$\lambda_1 < \lambda_k \ \ \forall k \geq 2\,. \tag{3}$$

For all $p \in \mathbb{N}$, we denote by $H^p(\Omega)$ the classical Sobolev space (so that $H^0(\Omega) = L^2(\Omega)$), and we recall (theoreme 4.3.25 of the textbook) that for $p > \frac{d}{2}$, $H^p(\Omega) \subset \mathcal{C}^0(\bar{\Omega})$. Here, we have denoted by $\mathcal{C}^0(\bar{\Omega})$ the space of continuous functions on $\bar{\Omega}$.

The following regularity theorem holds (for bounded regular open sets):

Théorème 1 *Let $p \in \mathbb{N}$, $f \in H^p(\Omega)$, and $u \in H_0^1(\Omega)$ the solution of*

$$\left[\begin{array}{l} -\Delta u = f \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{array}\right. \tag{4}$$

then $u \in H^{p+2}(\Omega)$.

Finally, the problem consists in finding solutions for an evolution partial differential equation. We will seek those solutions in the space $X = \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ of continuous functions from $[0,T]$ in $L^2(\Omega)$ which are also L^2 of $]0,T[$ with values in $H_0^1(\Omega)$. We recall that X is complete for the norm

$$||u||_X = \sup_{t\in[0,T]} ||u(t,\cdot)||_{L^2} + \left(\int_0^T ||u(t,\cdot)||_{H_0^1}^2\,dt\right)^{\frac{1}{2}}.$$

---

1. Show that $\forall k \geq 1,\ \phi_k \in \mathcal{C}^\infty(\bar{\Omega})$ and that $\forall k \geq 1,\ \lambda_k = \lambda(\phi_k) = \displaystyle\int_\Omega |\nabla\phi_k|^2\,dx\,.$

2. Gronwall's lemma. Let $b \in \mathcal{C}^0(\mathbb{R}^+)$ and $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ a $\mathcal{C}^1$ function which verifies

$$\forall t \geq 0,\ f'(t) \leq b(t) f(t)\,.$$

Show that

$$\forall t > 0,\ f(t) \leq f(0) \exp\left(\int_0^t b(s)\,ds\right)\,.$$

One may introduce the function g defined by

$$\forall t \geq 0,\ g(t) = f(t) \exp\left(-\int_0^t b(s)\,ds\right)\,.$$

3.1 We consider, for $f \in L^2(\Omega)$, the following problem

$$\left[\begin{array}{l} -\Delta u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{array}\right. \tag{5}$$

Recall the arguments which permit to show that (5) has a unique solution $\bar{u} \in H_0^1(\Omega)$.

3.2 Let $u_0 \in L^2(\Omega)$. We consider now the evolution problem

$$\left[\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x), \end{array}\right. \tag{6}$$

where f is the function of the preceding question. In particular f does not depend on time.

- Recall the result by which (6) admits a unique solution $u \in \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$.
- Show that the solution $u(t)$ of (6) converges exponentially in time in $L^2(\Omega)$ to $\bar{u}$ which is the solution of (5).

We wish to study an evolution equation similar to the one before, whose solution converges to an eigenfunction of the Laplacian. We propose the following equation in which u_0 satisfies now $||u_0||_{L^2(\Omega)} = 1$.

$$\left[\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta u = \mu(t) u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x), \\ ||u(t,.)||_{L^2(\Omega)} = 1,\ \forall t \geq 0. \end{array}\right. \tag{7}$$

Warning: pay attention to the fact that in the preceding equation μ is also an unknown of the problem.

The remaining part of the problem consists in showing that one can construct solutions to this equation. We show also that (in a particular case) those solutions do indeed converge to the first eigenfunction of the Laplacian ϕ_1.

---

4. A priori estimates. Let $T > 0$. We assume in this question only that there exists u solution of (7) on $[0, T]$, which is as smooth as desired. Show that

$$\mu(t) = \lambda(u(t)) = \int_\Omega |\nabla u|^2(t,x)\,dx\,. \tag{8}$$

Deduce that $\lambda(u(t)) \geq \lambda_1$ with equality if and only if $u(t,x) = \pm\phi_1(x)\ \forall x \in \Omega$. Show also that $\lambda(u(t))$ is a non increasing function on $[0, T]$.

Remark 2 *Therefore, in spite of appearances, (7) is in fact a non-linear equation.*

In order to show the non-linear nature of the equation, we rewrite (7) under the variational form:
Find $u \in \mathcal{C}^0(0,T; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega))$ satisfying

$$\left[\begin{array}{l} \forall\phi \in H_0^1(\Omega),\ \forall t \in [0,T], \\ < u(t), \phi > + \displaystyle\int_0^t a(u(s),\phi)\,ds =< u(0),\phi > + \int_0^t \lambda(u(s)) < u(s),\phi > \,ds \\ u(0,x) = u_0(x)\,. \end{array}\right. \tag{9}$$

We admit that this variational form is equivalent to the classical variational formulation corresponding to the problem

$$\left[\begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta u = \lambda(u(t))u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u(0,x) = u_0(x)\,. \end{array}\right. \tag{10}$$

And we will take from now on and for the rest of the problem $u_0 \in H_0^1(\Omega)$ (instead of L^2) satisfying $||u_0||_{L^2(\Omega)} = 1$.

---

5. Construction of particular solutions. In this question, we suppose that $u_0 \in E_N$ and we will write $u_0 = \sum_{i=1}^N \alpha_i^0 \phi_i$.

5.1

- Show that there exists $T^* > 0$ and a unique solution u of (9) satisfying $u(t,.) \in E_N$, $\forall t \in [0, T^*[$. (Write $u(t,x) = \sum_{i=1}^N \alpha_i(t)\phi_i(x)$, show that $\lambda(u(t)) = \sum_{i=1}^N \lambda_k \alpha_k(t)^2$ and write the differential system satisfied by $(\alpha_i(t))_{1\le i\le N}$.

- Show that $||u(t,\cdot)||_{L^2}^2 = \sum_{i=1}^N (\alpha_i(t))^2 = 1$ and deduce that the solution is actually global in time, that is to say $T^* = +\infty$.

- Show also that $u \in \mathcal{C}^\infty([0,+\infty[\times\bar{\Omega})$.

5.2 We suppose here that $\alpha_1^0 > 0$. Show that $\lambda(u(t))$ is a non increasing function of time which converges to λ_1 when t tends to $+\infty$. Show that $\alpha_1(t)$ is a non decreasing function of time which converges to 1 as t tends to $+\infty$.

6. Construction of solutions in the general case. We take now $u_0 \in H_0^1(\Omega)$ and we assume that

$$\int_\Omega u_0(x)\,\phi_1(x)\,dx > 0.$$

6.1 Show that there exists a sequence $(u_0^N)_{N\ge 1}$ such that

$$\begin{array}{l} \forall N \ge 1,\ u_0^N \in E_N\,, \\ \forall N \ge 1,\ ||u_0^N||_{L^2(\Omega)} = 1\,, \\ \forall N \ge 1,\ \int_\Omega u_0^N(x)\,\phi_1(x)\,dx > 0\,, \\ u_0^N \to u_0 \text{ when } N \to +\infty \text{ in } H_0^1(\Omega), \\ \big(\lambda(u_0^N)\big)_{N\ge 1} \text{ is bounded.} \end{array}$$

6.2 For all $N \ge 1$, we construct, thanks to question 5. a solution $u^N(t,x)$ of (9) with initial data u_0^N. Show that one has

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^N - u^M)^2\,dx + \int_\Omega |\nabla(u^N - u^M)|^2\,dx = \frac{\lambda(u^N) + \lambda(u^M)}{2}\int_\Omega (u^N - u^M)^2\,dx\,.$$

(You can use the identity $ab - cd = \left(\frac{a+c}{2}\right)(b-d) + \left(\frac{b+d}{2}\right)(a-c)$).

Deduce that there exist two constants $C_1 > 0$ and $C_2 > 0$ independent of N and M but possibly depending on T such that

$$\forall t \in [0,T],\ \int_\Omega (u^N(t,x) - u^M(t,x))^2\,dx \le C_1 \int_\Omega (u_0^N(x) - u_0^M(x))^2\,dx\,,$$

and

$$\int_0^T \int_\Omega |\nabla(u^N(t,x) - u^M(t,x))|^2\,dx\,dt \le C_2 \int_\Omega (u_0^N(x) - u_0^M(x))^2\,dx\,.$$

6.3 Deduce that $(u^N)_{n\geq 1}$ is a Cauchy sequence in $\mathcal{C}^0(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))$ which converges. We call $u^\infty \in \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ its limit.

Show finally that $(\lambda(u^N))_{N\geq 1}$ converges in $L^1(0,T)$ to $\lambda^\infty = \lambda(u^\infty)$ and that λ^∞ is a non increasing function.

Show that $||u^\infty(t)||_{L^2} = 1$, $\forall t \in [0,T]$.

6.4 Show that u^∞ is a solution of (9) with initial data u_0.

Problem 2 - Weyl's inequality (Green paper, 8 points)

The aim of the problem is to show a result (inequality (12)) which can be interpreted in quantum physics as the Heisenberg's uncertainty principle.

We call X the space of functions :

$$X = \Big\{ x(t) \in \mathcal{C}^{\infty}(\mathbb{R}_+) \text{ with real values : } \int_0^{+\infty} \Big[x^2(t) + t^2 x^2(t) + (\frac{dx}{dt})^2(t) \Big]\, dt < +\infty \Big\}.$$

1) Show that if $x \in X$ then $tx^2(t)$ tends to 0 when t tends to $+\infty$.

Indication : Integrate by parts

$$\int_0^t \tau x(\tau) \frac{dx}{d\tau}(\tau)\, d\tau.$$

2) By developing

$$\int_0^{+\infty} (\frac{dx}{dt} + tx)^2\, dt$$

and integrating by parts, show that for all $x \in X$,

$$\int_0^{+\infty} (\frac{dx}{dt})^2(t)\, dt \geq \int_0^{+\infty} (1 - t^2) x^2(t)\, dt. \tag{11}$$

3) Substituting $y(t/a)$ to $x(t)$ in (11) , show that there holds $\forall y \in X$

$$\int_0^{+\infty} (\frac{dy}{d\tau})^2(\tau)\, d\tau - a^2 \int_0^{+\infty} y^2(\tau)\, d\tau + a^4 \int_0^{+\infty} \tau^2 y^2(\tau)\, d\tau \geq 0, \quad \forall\, a > 0.$$

4) Deduce the inequality

$$\forall x \in X,\ \int_0^{+\infty} x^2(t)\, dt \leq C \Big(\int_0^{+\infty} t^2 x^2(t)\, dt \Big)^{1/2} \Big(\int_0^{+\infty} (\frac{dx}{dt})^2(t)\, dt \Big)^{1/2}, \tag{12}$$

with $C = 2$.

We now want to show that $C = 2$ is the smallest constant for which (12) holds. We consider the minimization problem over X

$$\inf \int_0^{+\infty} (\frac{dx}{dt})^2(t)\, dt \tag{13}$$

under the so-called isoperimetric equality constraint

$$\int_0^{+\infty} (t^2 - 1)\, x^2(t)\, dt = -1. \tag{14}$$

In order to find its solution, we introduce the Lagrangian

$$\mathcal{L}(x,\lambda) = \int_0^{+\infty} (\frac{dx}{dt})^2(t)\,dt + \lambda\Big[\int_0^{+\infty} (t^2-1)\,x^2(t)\,dt + 1\Big].$$

5) For $h \in X$ compactly supported in $[0,+\infty[$, compute the derivative of the Lagrangian $\mathcal{L}$ at x applied to h, that is to say the quantity

$$\lim_{\varepsilon\to 0} \frac{\mathcal{L}(x+\varepsilon h,\lambda) - \mathcal{L}(x,\lambda)}{\varepsilon}.$$

6) Show that the necessary optimality conditions are given by the Euler-Lagrange equation

$$-\frac{d^2x}{dt^2} + \lambda(t^2-1)x = 0 \tag{15}$$

and the transversality condition

$$\frac{dx}{dt}(0) = 0. \tag{16}$$

7) We recall that

$$\int_0^{+\infty} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^{+\infty} t^2 e^{-t^2}\,dt = \frac{\sqrt{\pi}}{4}.$$

Verify that $x_0(t) = \frac{2}{\pi^{1/4}} e^{-t^2/2}$ satisfies (15) for $\lambda = 1$, the transverslity condition (16) and the isoperimetric constraint (14). Compute $\int_0^{+\infty} (\frac{dx_0}{dt})^2(t)\,dt$, and show that x_0 is a solution of the minimization problem (13)-(14).

8) Show that the smallest constant C is equal to 2.

Indication : One can show for instance using (11) that if C_{opt} is the smallest constant, then

$$\frac{1}{C_{\text{opt}}^2} \le \frac{1}{4}\int_0^{+\infty} (\frac{dy}{dt})^2\,dt$$

for all $y \in X$ which satisfies the isoperimetric constraint (14).