Preface

The topic of this book is homogenization theory and its applications to optimal design in the conductivity and elasticity settings. Its purpose is to give a self-contained account of homogenization theory and explain how it applies to solving optimal design problems, from both a theoretical and a numerical point of view. The application of greatest practical interest targeted by this book is shape and topology optimization in structural design, where this approach is known as the *homogenization method*.

Shape optimization amounts to finding the optimal shape of a domain that, for example, would be of maximal conductivity or rigidity under some specified loading conditions (possibly with a volume or weight constraint). Such a criterion is embodied by an *objective function* and is computed through the solution of a *state equation* that is a partial differential equation (modeling the conductivity or the elasticity of the structure). Apart from those areas where the loads are applied, the shape boundary is always assumed to support Neumann boundary conditions (i.e., isolating or traction-free conditions). In such a setting, shape optimization has a long history and has been studied by many different methods. There is, therefore, a vast literature in this field, and we refer the reader to the following short list of books, and references therein [39], [42], [130], [135], [149], [203], [220], [225], [237], [245], [258]. We shall not consider the case of Dirichlet boundary conditions (for which the homogenization theory is somehow very different), referring the reader only to [62], [63], [64], [267], and references therein. Nor shall we discuss any other types of optimal design problems (like material properties optimization), even in the cases where homogenization could be relevant. We note that optimal design may be seen as a special branch of *optimal control*, where the control is the domain itself or the coefficients in the domain (the material properties). There are also many relevant references in optimal control, and more specifically in controllability, of so-called *distributed systems* (i.e., for which the state equation is a partial differential equation). In addition to the pioneering work of Pontryagin [223], we simply mention the classical books of Lions [164], [165].

In order to shed light on the originality of the homogenization method, we first briefly recall what can be called the *classical method for shape op-*

timization. The main idea goes, at least, as far back as Hadamard [129], and has been further developed by many authors, e.g., [39], [72], [203], [220], [255], [258]. Given an initial guess of the shape, one allows the boundary to vary along its normal (or, more generally, along any smooth velocity field). From both a theoretical and numerical pont of view, this approach gives a convenient framework for applying the calculus of variations. Nevertheless, the method suffers from two main drawbacks: It requires a smooth parametrization of the boundary, and it never changes the topology of the original domain (i.e., its number of holes or connected components). In structural design, this is often a serious drawback, since it is widely acknowledged that a porous structure of the same weight as a bulky one may be drastically more efficient. There have been some recent attempts to change the topology in this framework of shape sensitivity (e.g., the buble method [103], or the topology gradient [68], [257]), but they are either ad hoc methods, or very close in spirit to the homogenization method. Let us mention in passing that topology optimization can also be studied by stochastic approaches, like, for example, genetic algorithms, simulated annealing, or neural networks (e.g., [114], [143]).

The homogenization method was developed in order to remove smoothness and topological constraints inherent in the classical method of shape optimization. Another major difference of the homogenization method is the class of numerical algorithms it yields. As classical numerical methods are *shape-tracking* algorithms (where the shape fits to the mesh, which is deformed during the iterations), the homogenization-based numerical methods are rather *shape-capturing* algorithms (where the shape is captured on a fixed mesh). Before going to numerical issues, we now describe the main features of the homogenization approach.

In a first step, we rephrase a shape optimization problem as a two-phase optimization problem. Thanks to the homogeneous Neumann boundary conditions that they support, the holes in the domain may be filled by a very weak material (a poor conductor or a compliant elastic phase). In the limit where the properties of this ersatz material go to zero, we recover holes with Neumann boundary conditions (note that such a limit process is not always easy to justify rigorously). In other words, shape optimization is just the degenerate limit of two-phase optimal design, where the interface between the two phases replaces the boundary of the shape. A precise definition of two-phase optimization is as follows. In the conductivity setting, given two conducting materials, the goal is to find the best arrangement of the materials in a fixed domain that maximizes, for example, overall conductivity.

Similarly, in the elasticity setting, the goal is to find the best way of mixing two elastic materials yielding the most rigid structure. In both cases, there is a volume constraint on the best component phase, which may be seen as the extra cost on the good conductor or the stiff material (which are expected to be more expensive). Two-phase optimization is also an optimal design problem of interest, and we study it in great detail.

A key feature of two-phase optimal design problems is that they usually do not admit a solution in the absence of any smoothness or topological constraint on the interface or on the shape boundary. The root of this pathology is that it is often more advantageous having many tiny inclusions of one phase into the other than just a few big ones. Therefore, any proposed design can be improved by making small variations of the phases arrangement, and cannot be optimal. This process of making increasingly fine inclusions (or holes) shows that optimality can be achieved only in the limit of infinitely fine scale mixture of the two phases. Such mixtures are called *composite materials* and allow one to define optimal generalized designs. Homogenization is precisely the right mathematical tool for studying composite materials and defining generalized optimal designs.

Therefore, in a second step we introduce generalized designs for which one can prove existence of an optimum and derive necessary conditions of optimality. This process of enlarging the space of admissible designs in order to make the problem well-posed is called *relaxation*. A generalized design is often equivalently called a *relaxed*, or *composite*, design. Homogenization theory is the main ingredient for performing the relaxation of two-phase optimal design problems. As such, homogenization appears as just a trick for proving existence theorems, but its importance goes far beyond that, since it yields new numerical algorithms. Eventually, it remains to justify this relaxation process in the degenerate limit of shape optimization, and to post-process numerically these optimal composite designs to recover classical shapes.

The homogenization method in optimal design was initiated by Murat and Tartar in the late 1970's. Many joined their efforts in the development of that theory, and the first relevant references are the works of Murat and Tartar [205], [206], [269], [271] in France; Cherkaev, Lurie and their coworkers [115], [172], [178], [177], as well as Raitum [227] in Russia; and Kohn and Strang [152] in the United States. Surprisingly enough, the numerical algorithms based on the homogenization method have matured slowly. After the early contributions of Gibiansky and Cherkaev [115], Glowinski [117], Goodman, Kohn, and Reyna [119], and Lavrov, Lurie, and Cherkaev [160], which were mostly restricted to academic problems, it was the paper of Bendsoe and Kikuchi [47] in the late 1980's, which was the first to demonstrate the efficiency of such methods in shape optimization. After that pioneering contribution, many other works appeared that contributed to make the homogenization method one of the most popular and efficient methods for shape and topology optimization (see Chapter **??** and the References).

Although the main motivation of this book is shape optimization, while homogenization is, to some extent, just a technical tool for implementing optimization methods, we somehow proceed in reverse order. Indeed, the first chapter is concerned solely with homogenization theory. Mathematically, it can be defined as a theory for averaging partial differential equations. Homogenization has many potential applications, but we consider it only as a tool for deriving macroscopic or effective properties of microscopically heterogeneous media. As such it provides a firm basis to the notion of composite material obtained by mixing, on a very fine scale, several phase components. Although this question of averaging and finding effective properties is very old in physics or mechanics, the mathematical theory of homogenization is quite recent, going back to the 1970's. We expose the most general framework, known as the H-convergence, or G-convergence, introduced by Spagnolo [260], [261], and generalized by Tartar and Murat [204], [270].

In the framework of homogenization theory, the second chapter focuses on the study of two-phase composite materials. These composite materials (obtained as fine mixture limits of the two original phases) play a key role in the homogenization method used for shape optimization: Namely, as generalized designs. It is therefore crucial to find the range of all possible effective properties of such composites. This is called the *G-closure problem*, and it was solved in the conductivity setting for two isotropic phases by Murat and Tartar [205], [274], and Lurie and Cherkaev [175], [176]. Unfortunately, a similar answer is yet unknown in the elasticity setting. In this latter case, one can only obtain bounds that must be satisfied by the effective properties. In their most general form such bounds are called *Hashin-Shtrikman energy bounds* since they are derived by using the famous variational principle introduced by Hashin and Shtrikman [133]. They turn out to be optimal, i.e., the values of these bounds are exactly attained by special choices of composites. In particular, optimal composites can be chosen in the class of so-called *sequential laminates*, which will play an important part in shape optimization.

The third chapter is devoted to the application of homogenization to problems of two-phase optimization in conductivity. After introducing a precise mathematical framework, we give explicit examples of the nonexistence of optimal designs in the original space of "classical" designs. It is therefore necessary to relax the problem by enlarging the space of admissible designs, i.e., by allowing for generalized designs that are made up of composite materials. The key tool of this relaxation process is homogenization theory. As an additional advantage, the relaxed formulation allows one to derive optimality conditions that are at the root of numerical algorithms. Our exposition follows the original work of Murat and Tartar [205].

The fourth chapter is also concerned with the homogenization method for optimal design, but in the elasticity setting. Since the G-closure set of two elastic isotropic phases is still unknown, a rigorous relaxation procedure is available only for special objective functions including compliance (i.e., the work done by the loads). In such a case, the optimization problem being self-adjoint, the relaxation requires only the knowledge of optimal energy bounds instead of the full G-closure. Those bounds (called *Hashin-Shtrikman bounds*) are precisely studied in Chapter **??**, where they are shown to be attained by sequential laminates. Therefore, in the relaxation process the unknown full G-closure set can be replaced by its explicit subset of sequential laminates. Eventually, a complete relaxation procedure for shape optimization (i.e., when one of the phases degenerates to holes) is rigorously established. Here we follow mainly our work [15], [21], that of Gibiansky and Cherkaev [115], [116], and that of Kohn and Strang [152].

Finally, the fifth chapter is devoted to numerical issues for the homogenization method in optimal design. We mostly focus on shape optimization for elastic structures, since it is by far the most important application of the homogenization method. Building upon our knowledge of the relaxed, or homogenized, formulation, we discuss the two main types of algorithms, the optimality criteria and gradient methods, as well as several numerical technicalities. In particular, we explain how a numerical procedure, called *penalization*, allow one to post-process the optimal generalized designs in order to recover classical shapes. This chapter is illustrated by several results of two-dimensional and three-dimensional computations. Our exposition is complementary to that of Bendsoe in his book [42], which is a very good introduction to the numerical aspects of the homogenization method for a more practically inclined reader.

In the sequel, we do not point out open problems, but it is clear that the range of applications covered by this book, although very important, is somehow narrow. Of course, there are many other types of optimal design problems, apart from shape optimization, that have not yet been attacked with the homogenization method (or that cannot be treated by this method). It is our hope that this book can serve as a basis for further developments in new directions. The references are not exhaustive, by any means, and so I apologize to those inadvertently left off of the list.

To conclude this introduction, I want to thank all my coworkers, with whom I learned so much about shape optimization: S. Aubry, Z. Belhachmi, E. Bonnetier, G. Francfort, F. Jouve, R. Kohn, V. Lods, and F. Murat. I benefited from many stimulating discussions with G. Milton and L. Tartar, that I acknowledge with much pleasure. Special thanks are due to F. Jouve, who was the corner stone of the development of a three-dimensional shape optimization code, and to G. Francfort for his comments on an earlier version of the manuscript. Last but not least, I owe a lot to F. Murat, who had so much influence on my understanding of homogenization, and to R. Kohn, who introduced me to optimal design.

Paris, France
May 2001

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