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Optimal design for minimum weight and compliance in plane stress using extremal microstructures

G. ALLAIRE *, ** and R. V. KOHN ***, ****

ABSTRACT. — We study the shape optimization of a two-dimensional elastic body loaded in plane stress. The design criteria are compliance and weight. A relaxed formulation is used, whereby perforated composite materials are admitted as structural components. This approach has the advantage of placing no implicit restriction on the topology of the design. A similar method has recently been explored by Bendsoe, Kikuchi, and Suzuki, with the goal of "topology optimization". Our work differs from theirs in two important respects: we emphasize the use of optimal microstructures, and we use a formulation based on stresses. While our numerical work is two dimensional, the method could in principle also be applied to three dimensional structures.

Contents

1. Introduction

The basic problem of structural optimization is to choose the shape or composition of a structure so as to optimize some feature of its behavior. This paper is concerned

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* Commissariat à l'Énergie Atomique, LETR/SERMA/DMT, C.E.N. Saclay, 91191 Gif-sur-Yvette, France.

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*** Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, U.S.A.

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primarily with shape optimization, in the context of two-dimensional elastic structures loaded in plane stress. Our goal is to minimize the *weight* (area) subject to a constraint on the *compliance* (work done by the load).

There is by now an extensive literature on shape optimization; see for example the reviews ([Ding, 1986]; Haftka & Grandhi, 1986]; [Pironneau, 1984]; [Venkayya, 1978]) and conference proceedings ([Bennett & Botkin, 1986]; [Mota Soares, 1987]). Virtually all this work implements some variation of the following idea, which we shall call the "standard approach." One begins with an initial design in which part of the boundary is free, *i.e.* subject to optimization. Its configuration is determined by finitely many parameters, typically the positions of certain control nodes. The elasticity problem is discretized, usually using finite elements or a boundary integral formulation. The task of optimization then becomes a large nonlinear programming problem for the positions of the control nodes. It can be solved by a version of steepest descent, based on successive sensitivity analyses. Other optimization techniques are also sometimes used, including the method of optimality criteria (e. g. [V, 1978]) and sequential linear programming (e. g. [Fleury & Braibant, 1986]).

The main drawback of this approach is the dependence of the "optimal" design on the form of the initial guess. As with any nonlinear programming problem there is the difficulty of local minima. Worse, however, is the fact that the topology of the design is fixed in the very formulation. This is a severe limitation, since the topology greatly influences the performance of the "optimal" design (*see* e. g. [Bendsoe & Rodrigues, 1992], [Olhoff *et al.*, 1991]).

There have been some attempts to permit change of topologiy in an otherwise "standard" optimization code, notably [Atrek, 1989] and [Eschenauer *et al.*, 1992]. Recently, however, a totally different alternative has emerged. It is the use of a "relaxed formulation". The essential idea of relaxation is to admit perforated composite materials as structural components, along with the originally given material. The original design problem is *not* being changed; we simply permit a microstructure of many small holes rather than insisting upon a few large ones. After relaxation, the task of structural optimization looks more like a sizing problem than one of shape optimization. Since the introduction of perforated composites "looks like" an expansion of the design space, it tends to destroy local minima and to yield better performance for a given computational mesh.

This new approach has evolved through the joint (often independent) effort of many individuals. A sampling of the literature includes ([Bendsoe & Kikuchi, 1988]; [Cheng & Olhoff, 1981]; [Gibiansky & Cherkaev, 1984]; [Kohn & Strang, 1986]; [Lurie *et al.*, 1982]; [Lurie & Cherkaev, 1986]; [Murat & Tartar, 1985]; [Olhoff *et al.*, 1981]; [Raitum, 1979], and [Rozvany *et al.*, 1987]. See also the recent conference proceeding volume [Bendsoe & Mota Soares, 1992]. There are situations (*see* for example [Goodman *et al.*, 1986]; [Kawohl *et al.*, 1991], and [K & S, 1986]) in which a relaxed formulation is required for the very existence of an optimal design, if one desires a global optimum with no topological constraints. In the context of shape optimization the situation is as follows: one could consider an optimal design first with one hole, then with two, and so forth. As the number of holes gets larger, the performance gets better. In the limit of infinitely many holes one obtains a global optimum which is not a conventional design at all. Rather, it is a structure made from composite materials obtained by perforation.

The relaxed formulation is convenient for proving the existence of optimal designs, and for deriving necessary conditions of optimality. But its importance goes far beyond that. The relaxed formulation provides a method for actually computing optimal designs without placing implicit restrictions on the topology. The resulting structures might not be easy to manufacture, since they will make use of fine-scale performation. But they can nevertheless serve as benchmarks against which to compare any design. They may also suggest a good initial guess for a conventional optimization code. Striking examples are provided by the recent papers [B & R, 1992] and [O *et al.*, 1991]: by using a relaxed formulation as a pre-processor for a more conventional code, they obtain a dramatic improvement in the performance of a computed "optimal" design.

The numerical calculation of relaxed optimal designs for elastic structures has recently been considered by a number of authors: ([Bendsoe, 1989]; [B & K, 1988]; [Diaz & Bendsoe, 1992]; [Jog *et al.*, 1992]; [Suzuki & Kikuchi, 1991]). Some of these individuals view it as a method of "topology optimization." Our approach differs from most of the work just cited in two important respects. First, we use *optimal* composites rather than ones chosen in some ad-hoc way. And second, we use a *stress-based*, variational formulation. These two choices make the relaxed design problem resemble the analysis of elastostatic equilibrium for a special, physically nonlinear material [*see* (2.16) below].

The use of extremal composites for the optimal design of plates was considered some time earlier by Gibiansky and Cherkaev [G & C, 1984]. That paper is roughly the analogue of the present work in the context of plate theory. Their more recent paper [Gibiansky & Cherkaev, 1987] includes a description of the optimal composites at a given stress, for mixtures of two isotropic elastic materials in both two and three space dimensions. It thus overlaps considerably with Sections 4 and 7 of the present work. However, our method is substantially different and in our opinion more systematic than that of [G & C, 1987].

It is obvious that the performance of a structure depends both on its shape and on its composition; *see* e.g. [Ashby, 1991] for a thoughtful analysis. In considering relaxed design problems, we recognize that these two dependencies are really one and the same. The use of extremal microstructures is actually just shape optimization at a microscopic length scale.

The stress-based, variational method we use here originated in [K & S, 1986]. That paper actually computed our relaxed functional (*i.e.* evaluated the optimal composites) for the special case of an elastic material with Poisson's ratio zero. It used a method called "polyconvexification". The approach used here, based on effective moduli of composites, is physically more intuitive.

Our attention is focussed on two dimensional structures in plane stress. However it should also be possible to optimize three dimensional structures using a similar technique. The characterization of the optimal microstructures is more complicated than in 2D, but otherwise similar. We carry out the details in Section 7. The crucial bounds on a mixture of two isotropic elastic materials are also worked out in [G & C, 1987]. The numerical minimization of the relaxed 3D functional requires solving an extremely large, nonsmooth mathematical programming problem. We have not yet had the courage to undertake this.

2. The relaxed design problem

This section presents the compliance/weight optimization problem in both its classical and relaxed formulations. Our discussion applies equally to the 2D and 3D settings. Various technical issues – for example, an explanation of what we mean by a "perforated composite" – are postponed until Section 3.

Shape optimization for minimum compliance and weight is a standard model problem, widely studied in the literature on structural optimization. Its motivation lies in the status of compliance as a global measure of rigidity. The problem can be formulated (prior to any relaxation) as follows. We begin with a region $\Omega\subset\mathbf{R}^n$ ($n=2$ or 3), occupied by a linearly elastic material with Hooke's law A_0. We suppose that Ω is loaded on its boundary by a known function $f:\partial\Omega\to\mathbf{R}^n$. (Other boundary conditions are also possible, e.g. part of $\partial\Omega$ might have a specified displacement.) We intend to remove a subset $H\subset\Omega$, consisting of one or more holes; the new boundaries created this way will be traction-free. The equations of elasticity for the resulting structure are:

$$
(2.1)\qquad \begin{cases} \sigma=A_0\,e(u), \qquad e(u)=\dfrac{1}{2}(\nabla u+\nabla u^{T}) \\ \operatorname{div}\sigma=0 \quad \text{in } \Omega\setminus H \\ \sigma.n=f \quad \text{at } \partial\Omega, \qquad \sigma.n=0 \quad \text{at } \partial H, \end{cases}
$$

and the compliance is

$$
(2.2)\qquad c(H)=\int_{\partial\Omega} f.u=\int_{\Omega\setminus H}\langle A_0\,e(u),\,e(u)\rangle.
$$

Our goal is to minimize the weight subject to a constraint on the compliance:

$$
(2.3)\qquad \min_{c(H)\leqq M}\ |\Omega\setminus H|,
$$

where M is a constant. One may just as well minimize the compliance subject to a constraint on the weight:

$$
(2.4)\qquad \min_{|\Omega\setminus H|\leqq M'}\ c(H).
$$

The equivalence of (2.3) and (2.4) is a reflection of the fact that removing material always increases the compliance.

Following standard practice, we shall handle the constraint by means of a Lagrange multiplier. Thus instead of (2.3) or (2.4) we shall actually focus on the unconstrained problem

$$\min_{\mathrm{H}} [c(\mathrm{H}) + \lambda |\Omega \setminus \mathrm{H}|]. \tag{2.5}$$

Here λ is a positive constant, which can be viewed as a Lagrange multiplier for the constraint in (2.4). [Equivalently, λ^{-1} can be viewed as a Lagrange multiplier for the constraint in (2.3).] It is easy to see that any solution of (2.5) also solves (2.3) and (2.4) for suitably chosen $\mathrm{M} = \mathrm{M}(\lambda)$ and $\mathrm{M}' = \mathrm{M}'(\lambda)$. It is easy to see that as λ increases from 0, the weight of the optimal design decreases and its compliance increases. Thus $\lambda \mapsto \mathrm{M}(\lambda)$ and $\lambda \mapsto \mathrm{M}'(\lambda)$ are monotone functions. (We believe that they are continuous in λ, but this has not been proved.)

Some care is in order concerning the meaning of the term "solution". We believe that as stated, these problems actually *have no "classical" solutions* for most choices of the data. Analogous assertions have been proved for several closely related problems ([K *et al.*, 1991]; [K & S, 1986]; [M & T, 1985]). Physically, the point is that it may be advantageous to use many small holes rather than a few large ones. Achieving the optimum may require a limiting procedure involving infinitely many, infinitely fine holes. In other words, the optimal behavior may be achieved only by a "generalized" design consisting of composite materials made by perforation. The microstructure of the optimal "generalized" design will vary from point to point, depending on the data of the problem. (A rigorous discussion of what we mean by a composite material will be found in Sec. 3.)

As just presented, the use of composite materials might appear to be just a trick for proving existence theorems. In fact its importance goes much further. The use of composites permits one to calculate optimal designs without making hidden or implicit topological assumptions. The primary goal of the present work is to demonstrate this assertion.

The crucial first step, for both theory and calculation, is a precise formulation of what we mean by a "generalized" design. Such a structure is determined by two functions: its effective Hooke's law $\mathrm{A}(x)$, taking values in the space of fourth-order tensors; and its local volume fraction $\theta(x)$, taking values between 0 and 1. The equations of elasticity become

$$\begin{cases} \sigma = \mathrm{A}(x)\, e(u) \\ \operatorname{div} \sigma = 0 \quad \text{in } \Omega, \\ \sigma . n = f \quad \text{at } \partial\Omega, \end{cases} \tag{2.6}$$

and the compliance is defined as

$$c[\mathrm{A}] = \int_{\partial\Omega} f.u = \int_{\Omega} \langle \mathrm{A}(x)\, e(u), e(u) \rangle. \tag{2.7}$$

The weight is proportional to the total amount of material, which is the integral of the local volume fraction $\theta(x)$. Therefore our minimization problem (2.5) becomes

$$\min_{\theta(x),\ A(x)} \quad c[A]+\lambda \int_\Omega \theta(x)\,dx. \tag{2.8}$$

The process of permitting composites as generalized designs is called "relaxation", so we refer to (2.8) as the relaxed formulation of the design problem. Notice that the classical designs are included in the relaxed formulation: a hole is obtained by taking $\theta(x)=0$ and $A(x)=0$ on some subset $H\subset\Omega$.

This process of relaxation has been discussed by many authors. In the present context, the main points are these:

(2.9 *a*) *The relaxed problem* (2.8) *is equivalent to the original one* (2.5). *Specifically, their minimum values are equal, and every solution of* (2.8) *determines a minimizing sequence of classical designs for* (2.5).

(2.9 *b*) *The relaxed problem can be reformulated as a nonlinear optimization over the class of statically admissible stresses. It is thus amenable to numerical minimization via the finite element method.*

(2.9 *c*) *The relaxed problem always has a solution, at least in* 2 *D*.

We shall explain (2.9 *a*) and (2.9 *c*) in Sec. 3. (They are virtually tautologies, once the term "composite material" is interpreted properly.) Most of the rest of this paper is devoted to assertion (2.9 *b*).

We come now to a subtle point: to determine the relaxed design problem completely, we must specify the admissible values of $\theta(x)$ and $A(x)$ in (2.8). In other words, we must specify precisely which composite materials are to be considered as design components. The most general choice is to permit *all composites attainable by perforation*. This means that we require only

$$0\leqq\theta(x)\leqq 1 \quad \text{and} \quad A(x)\in G_{\theta(x)} \tag{2.10}$$

for each $x\in\Omega$, where

(2.11) G_θ = the set of all effective Hooke's laws describing composites obtainable from the original elastic material A_0 via perforation with volume fraction $1-\theta$ of holes.

[*See* Sec. 3 for further discussion of (2.10)-(2.11).] This is the choice we prefer, because it leads naturally to the fundamental properties (2.9 *a-c*) above. Other choices are also possible. For example, one can restrict attention to the class of sequentially laminated composites ([Avellaneda, 1987]; [B, 1989]; [J *et al.*, 1992]). Since extremal composites can be taken to be sequentially laminated (*see* Sec. 4), this is operationally equivalent to (2.10)-(2.11). Another alternative is the choice of Bendsoe, Diaz, Kikuchi, and Suzuki: they work with periodic arrays of rectangular holes, characterized (in 2 D) by three parameters – the volume fraction, the aspect ratio, and an orientation angle. Through more elementary than ours, this approach has the defect of being arbitrary: why should rectangular holes be better, say, than elliptical ones? The complementary defect of our approach is that the set G_θ is not explicit: further analytical work is required to do the optimization over microstructures.

It remains to explain how, as a practical matter, one is to solve (2.8). This is an optimal control problem of a fairly standard type, except for one important catch: we do not know the precise form of the set of possible Hooke's laws G_θ. However, we do have partial knowledge of G_θ ([Allaire & Kohn, 1992*a* and *b*]; [A, 1987]; [Francfort & Murat, 1986]; [Kohn & Lipton, 1988]; [Milton & Kohn, 1988]). In particular, we know how to minimize the complementary energy $\langle A^{-1}\tau, \tau\rangle$ over $A \in G_\theta$, for any given τ. (This will be explained in Sec. 4.)

We now show how (2.8) can be solved using only such partial information about G_θ. The first step is to represent the compliance variationally, using the principle of minimum complementary energy:

$$c[A] = \min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \int_\Omega \langle A(x)^{-1}\tau, \tau\rangle. \tag{2.12}$$

With this substitution, (2.8) becomes

$$\min_{\substack{0\le\theta(x)\le 1\\ A(x)\in G_{\theta(x)}}} \min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \int_\Omega [\langle A(x)^{-1}\tau, \tau\rangle + \lambda\theta(x)]\,dx. \tag{2.13}$$

The next step is simple but fundamental: we interchange the order of minimization in (2.13) to get

$$\min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \min_{\substack{0\le\theta(x)\le 1\\ A(x)\in G_{\theta(x)}}} \int_\Omega [\langle A(x)^{-1}\tau, \tau\rangle + \lambda\theta(x)]\,dx. \tag{2.14}$$

The third step is to put the minimization over $\theta(x)$ and $A(x)$ inside the integral. This is permissible because $\theta(x)$ and $A(x)$ are subject only to *algebraic* constraints (unlike τ, which must satisfy $\operatorname{div}\tau = 0$). We thus obtain

$$\min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \int_\Omega \Big(\min_{0\le\theta(x)\le 1} \ \min_{A(x)\in G_{\theta(x)}} \langle A(x)^{-1}\tau, \tau\rangle + \lambda\theta(x)\Big)\,dx. \tag{2.15}$$

This has the form of a (nonconvex) variational problem

$$\min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \int_\Omega F_\lambda(\tau)\,dx \tag{2.16}$$

with integrand

$$F_\lambda(\tau) = \min_{0 \leqq \theta \leqq 1} \min_{A \in G_\theta} [\langle A^{-1}\tau, \tau\rangle + \lambda\theta]. \tag{2.17}$$

The variational problem (2.16) *is the central result of this section, and the basis of everything that follows.* The point is that we can evaluate $F_\lambda(\tau)$ *without* complete knowledge of G_θ. It suffices to know just the "optimal lower bound on complementary energy"

$$\min_{A \in G_\theta} \langle A^{-1}\tau, \tau\rangle = f(\theta, \tau) \tag{2.18}$$

as a function of θ and τ. Given $f(\theta, \tau)$, evaluating (2.17) requires just a one-dimensional optimization in θ.

The upshot is this: in its original form (2.8), the relaxed design problem appears uncomputable for lack of knowledge of G_θ. But it has the equivalent variational formulation (2.16), which *is* computable. In its form, (2.16) looks like a problem of physically nonlinear (but geometrically linear) elasticity. It minimizes *compliance* + λ . *volume* over all statically admissible stress fields τ. Given the solution $\tau^*(x)$, one recovers an optimal design by reversing the argument that led from (2.8) to (2.16). It has volume fraction $\theta^*(x)$ of material and Hooke's law $A^*(x)$, where $\theta^*(x)$ and $A^*(x)$ achieve the minimum in (2.17) with $\tau = \tau^*(x)$. Since $A^*(x) \in G_{\theta^*(x)}$, there is a perforated composite with volume fraction $\theta^*(x)$ that has effective Hooke's law $A^*(x)$. This yields the microstructure of the optimal design at x. If we take the length scale of the microstructure to be small but finite, we get a "classical" design with many small holes whose behavior is essentially optimal.

The relaxed variational problem (2.16) is an improvement over the classical formulation (2.5) in several important ways. For one thing, it is actually easier to solve numerically. One can use a more or less standard version of the finite element method. The computational mesh remains fixed: there is no need for "front-tracking" to follow free boundaries. Thus, numerical discretization of (2.16) does not introduce hidden topological restrictions upon the form of the solutions. Another advantage is the *existence* of a solution. [We shall prove that (2.16) achieves its minimum in 2 D − *see* Sec. 3.] This is a matter of practical as well as theoretical interest: it suggests that a numerical method based on (2.16) should be stable under refinement of the computational mesh.

The passage from the relaxed formulation (2.8) to the relaxed variational problem (2.16) is elementary. But it is also fundamental. The main idea goes back to [K & S, 1986]. We are using quite strongly the choice of compliance as the specific design criterion: a similar argument would not be possible, for example, if the goal were to minimize $\int_\Omega |u|^2$.

The equivalence of (2.13) and (2.14) bears further examination. The former reflects the traditional formulation of the problem: one fixes a design, then solves an elasticity problem, then adjusts the design to improve its performance. The other order of minimization, (2.14), has a totally different interpretation. It fixes a candidate (statically admissible) stress field, then finds the optimal design for this stress, then adjusts the stress to achieve kinematic admissibility. The two are equivalent, because when minimizing a function of several variables the order of minimization is unimportant.

The passage from (2.14) to (2.15) depends, as we noted, on the fact that $\theta(x)$ and $A(x)$ have only local constraints, as expressed by (2.10)-(2.11). This physically intuitive assertion is not at all obvious from the mathematical viewpoint. Its justification rests upon the "local character of G-closure" [Dal Maso & Kohn, in preparation]. (*See* Sec. 3 for further discussion.)

It is important to mention that optimal designs are not in general unique. Indeed, nonuniqueness arises at both the microscopic and macroscopic length scales. By microscopic nonuniqueness, we mean that a given effective Hooke's law $A \in G_\theta$ can be achieved by several different microstructures (*see* Remark 4.1). By macroscopic nonuniqueness, we mean that the solution of the relaxed problem (2.16) is not necessarily unique. This is not just a matter of several local minima: for special boundary conditions there can actually be infinitely many, globally optimal designs (*see* Sec. 8)! In the presence of such nonuniqueness, a numerical method must inevitably incorporate some implicit selection mechanism. Our method seems to choose a design whose macroscopic characteristics vary smoothly rather than abruptly. The method of Bendsoe-Kikuchi-Suzuki seems to make a different choice (*see* Sec. 8).

Apart from mathematical technicalities (to be addressed in Sec. 3), we are left with two main tasks. The first is to evaluate the relaxed integrand $F_\lambda(\tau)$ explicitly, and the second is to solve (2.16) numerically. These are addressed for the 2D problem in Sections 4 and 5, which form the heart of this paper. Section 6 discusses the relation between our approach and the theory of Michell trusses. Section 7 explains how a similar approach can be used for the optimization of 3D structures. Finally, Section 8 discusses the relation between our work and that of Bendsoe-Kikuchi-Suzuki, Gibiansky-Cherkaev, and Kohn-Strang.

3. Some technical issues

We have thus far been cavalier in the use of the term "perforated composite". The goal of this section is to place our work on a sound mathematical foundation. Readers who are not expert in homogenization may wish to skim this section, or even to skip it entirely on first reading.

There is a mathematically rigourous theory of composite materials, based on the notion of G-convergence, for structures made from two (or more) nondegenerate elastic materials. This theory is summarized in Section 3 A, along with its application to optimal design.

Our principal interest is shape optimization. This is not strictly speaking covered by the general theory, since it considers structures made by mixing a nondegenerate elastic law ("the originally given material") with a totally degenerate one (the "holes"). As a practical matter, we resolve this difficulty by treating the "holes" as compliant but nondegenerate inclusions. In other words, we optimize the structure first, *then* pass to limit in which the "holes" are degenerate. This procedure is physically and mathematically correct for compliance optimization – through not necessarily for other design objectives. Its justification is the focus of Section 3 B.

3 A. STRUCTURAL OPTIMIZATION USING TWO NONDEGENERATE MATERIALS

Let A_0 and A_1 be two (nondegenerate) Hooke's laws, *i. e.* quadratic forms on symmetric tensors satisfying

$$m|\xi|^2 \leqq \langle A_i \xi, \xi \rangle \leqq M|\xi|^2, \qquad i=0, 1, \tag{3.1}$$

with $0<m<M<\infty$. Consider a domain $\Omega \in \mathbb{R}^n$ $(n=2, 3)$, loaded on its boundary by traction f. A "structure made from A_0 and A_1" is characterized by a marker function $\chi(x)$, indicating the location of material A_0:

$$\chi(x) = \begin{cases} 1 & \text{in material } A_0 \\ 0 & \text{in material } A_1. \end{cases} \tag{3.2}$$

The local Hooke's law is

$$A(x) = \chi(x) A_0 + (1-\chi(x)) A_1, \tag{3.3}$$

and the elastic response is determined by solving

$$\begin{cases} \sigma = A(x) e(u) \\ \operatorname{div} \sigma = 0 \quad \text{in } \Omega \\ \sigma . n = f \quad \text{at } \partial\Omega. \end{cases} \tag{3.4}$$

Now consider a family of such structures, corresponding to different marker functions $\chi^j(x)$, $j=1, 2, 3, \ldots$. We may suppose (this is the interesting case) that the spatial length scale of the mixture tends to 0 as $j \to \infty$. Let σ^j, e^j and A^j be the stress, strain, and local Hooke's law of the jth stucture. We view the limiting behavior as being that of a "structure consisting of composite materials made by mixing A_0 and A_1." Its stress and strain are

$$\sigma^\infty(x) = \lim_{j \to \infty} \sigma^j(x), \qquad e^\infty(x) = \lim_{j \to \infty} e^j(x) \tag{3.5}$$

(in the weak L^2 topology). They are related by an *effective Hooke's law* $A^\infty(x)$,

$$\sigma^\infty(x) = A^\infty(x) e^\infty(x). \tag{3.6}$$

The effective Hooke's law is called the G-limit of the sequence $\{A^j\}$, written $A^j \xrightarrow{G} A^\infty$. The existence of the G-limit – *i. e.* the fact that σ^∞ must be linearly related to e^∞ – is the fundamental compactness theorem of G-convergence, *see* e.g. [Murat, 1978] or [Zhikov *et al.*, 1979]:

PROPOSITION 1. — *Let* $\{\chi^j\}$ *be any sequence of marker functions (taking only the values* 0 *and* 1), *and denote by* $A^j(x)$ *the associated Hooke's law* (3.3). *There is a subsequence with the following property: for any* L^2 *traction* f, *the associated stresses and strains converge (weakly in* L^2) *to limits* $e^\infty(x)$, $\sigma^\infty(x)$, *and the limits satisfy* $\sigma^\infty(x) = A^\infty(x)\, e^\infty(x)$. *The "effective Hooke's law"* $A^\infty(x)$ *satisfies* (3.1) *for each* x. *It depends on the "microstructure",* i.e. *on the sequence* $\{\chi^j\}$, *but not on the particular load* f.

We have stated this result only for two-component composites and for traction boundary conditions, because that is the focus of our interest here. The general compactness theorem is much more general – it permits any number of (nondegenerate, nonrigid) component materials, and displacement as well as traction boundary conditions.

The local volume fraction of the composite is determined by the limiting behavior of the marker functions χ^j: if

$$\theta(x) = \lim_{j\to\infty} \chi^j(x) \tag{3.7}$$

(in the weak* topology on L^∞) then the limiting structure consists of A_0 and A_1 in volume fractions $\theta(x)$ and $1-\theta(x)$, at $x \in \Omega$.

We repeat that the limiting structure is viewed as being made up of composite materials, possibly varying from point to point. This might seem at first a tautology, but in fact there is a subtle distinction. The general theory of G-convergence provides us only with tensor-valued *functions* $A^\infty(\cdot)$. We wish to interpret the pointwise value $A^\infty(x)$ as the effective behavior of a (homogeneous) composite material. This is justified by the following result [Dal M & K, in preparation]. (*See* also [Cabib & Dal Maso, 1988] for a similar result cast in somewhat different language.)

PROPOSITION 2. — *For each* θ, $0 \leqq \theta \leqq 1$, *there is a closed set of fourth-order tensors* G_θ, *"the G-closure of* A_0 *and* A_1 *with volume fractions* θ, $1-\theta$", *with the following properties:*

a) if $\theta(x)$ *and* $A^\infty(x)$ *arise as in* (3.5)-(3.7), *then* $A^\infty(x) \in G_{\theta(x)}$ *for a.e.* x,

b) if $\theta(x)$ *and* $A^\infty(x)$ *are measurable functions satisfying the restriction* $A^\infty(x) \in G_{\theta(x)}$ *a.e., then* $\theta(x)$ *and* $A^\infty(x)$ *do arise as in* (3.5)-(3.7) *for a suitably chosen sequence of structures* $\{\chi^j\}$.

Thus, the set of all possible limits $\{\theta(x), A^\infty(x)\}$ is *precisely* the set of all functions satisfying the restriction $A^\infty(x) \in G_{\theta(x)}$ almost everywhere. We view the elements of G_θ as Hooke's laws of (homogeneous) composites made from A_0 and A_1 in volume fractions θ and $1-\theta$. The dependence $\theta \mapsto G_\theta$ is continuous. It is obvious that in the extremes $\theta=0$ and $\theta=1$, $G_0 = \{A_1\}$ and $G_1 = \{A_0\}$.

So far we have placed no restriction on the spatial character of our composites. There is an extensive theory of spatially periodic composites, *see* e.g. [Bensoussan *et al.*, 1978]. Periodicity is convenient, because it leads to an "explicit" formula for A^∞ in terms of the solutions of certain "cell problems." The hypothesis of periodicity is also useful for proving bounds on effective moduli, *see* e.g. ([A & K, 1992 *b*]; [M & K, 1988]). As it turns out, the spatially periodic composites are dense in G_θ [Dal M & K, in preparation]:

PROPOSITION 3. — *For any* θ, $0<\theta<1$, *let* P_θ *be the set of all Hooke's laws of spatially periodic composites which mix* A_0 *and* A_1 *with volume fraction* θ, $1-\theta$. *Then* $P_\theta \subseteq G_\theta$, *and the closure of* P_θ *is precisely* G_θ.

Thus spatial periodicity represents no significant restriction. In particular, for proving bounds on effective moduli, it is sufficient to consider the spatially periodic case. (We emphasize, however, that approximating an arbitrary $A \in G_\theta$ by a spatially periodic $A' \in P_\theta$ might require a microstructure with a very complicated unit cell.)

We are now prepared to justify the fundamental assertions (2.9 *a-c*) for the nondegenerate analogue of shape optimization. Prior to relaxation our "structures" have local Hooke's laws of the form (3.3). The compliance is

$$c[A]=\int_{\partial\Omega} f.u=\int_{\Omega} \langle A(x)\, e(u),\, e(u)\rangle, \tag{3.8}$$

where u is the elastic displacement, obtained by solving (3.4). After relaxation our "structures" can have any volume fraction and Hooke's law $\theta(x)$, $A(x)$, subject only to the restriction $A(x) \in G_{\theta(x)}$. Notice that if $A^{\infty}(x)$ arises from a sequence $A^{j}(x)$ as in (3.3)-(3.4) then the elastic displacements u^j converge (weakly) to u^{∞}, so

$$c[A^j]=\int_{\partial\Omega} f.u^j \to \int_{\partial\Omega} f.u^{\infty}=c[A^{\infty}].$$

Thus (3.8) remains the formula for the compliance also after relaxation.

The relaxed design problem is equivalent to the original one, by Proposition 2. In particular their minimum values are equal, and any solution of the relaxed problem leads to a minimizing sequence for the original one. The relaxed design problem is easily transformed to a (nonconvex) variational problem of the form

$$\min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}} \int_{\Omega} F_{\lambda}(\tau)\, dx, \tag{3.9}$$

by arguing as in (2.12)-(2.16). Notice that Proposition 2 is crucial here, too: it is what justifies putting the minimum over θ and $A \in G_\theta$ inside the integral [*see* (2.15)]. We can prove that the relaxed problem has a solution by arguing as follows. Consider a minimizing sequence, *i.e.* a sequence of (relaxed) designs for which

$$c[A]+\lambda \int_{\Omega} \theta(x)\, dx \tag{3.10}$$

approaches its minimum value. By Proposition 2, there is a sequence of classical designs with essentially the same performance. Passing to a subsequence if necessary, the compactness theorem (Proposition 1) provides a G-limit which, with its associated volume fraction $\theta(x)$, achieves the minimum in (3.10). Proposition 2 assures us that $A^{\infty}(x) \in G_{\theta(x)}$, so this pair $A^{\infty}(x)$, $\theta(x)$ solves the relaxed problem.

3 B. Shape optimization

To model an elastic body of variable shape, we should take $A_1=0$ in the above. Then the equations of elasticity (3.4) become

$$(3.11)\qquad \left\{\begin{aligned} \sigma=A_0 e(u) &\quad \text{in } \Omega\setminus H\\ \sigma=0 &\quad \text{in } H\\ \operatorname{div}\sigma=0 &\quad \text{in } \Omega\setminus H\\ \sigma.n=f &\quad \text{at } \partial\Omega, \end{aligned}\right.$$

where $H=\{x:\chi(x)=0\}$ is the set of "holes." If H has a piecewise smooth boundary then (3.11) is equivalent to the equations of elastostatics on $\Omega\setminus H$, with a traction-free boundary condition at ∂H [*see* (2.1)]. Notice that in our formulation one cannot remove material at a loaded part of the boundary.

The theory of G-convergence cannot be used, however, in the limit $A_1 \to 0$. Mathematically speaking, this limit violates the condition (3.1). Physically, the problem is that the effect of a crack is totally different from that of an inifinitesimally thin elastic inclusion. Indeed, as the width of an elastic inclusion tends to zero its influence on the displacement vanishes – unlike a crack, across which the displacement can be discontinuous.

As an operational matter, we avoid this difficulty by treating the "holes" as being occupied by an infinitesimally compliant elastic material. To be more specific, we formulate the relaxed optimization problem as in Section 3 A, with $A_1 \neq 0$; then we pass to the variational formulation (3.9), by evaluating for each symmetric tensor τ the relaxed integrand

$$(3.12)\qquad F_\lambda(\tau)=\min_{0\leq\theta\leq 1}\ \min_{A\in G_\theta}\ [\langle A^{-1}\tau, \tau\rangle+\theta\lambda];$$

then we pass to the limit $A_1 \to 0$ in the formula for $F_\lambda(\tau)$. In other words, for shape optimization the relaxed integrand is still given by (3.12), with

(3.13) G_θ = the limit of the G-closure of A_0 and A_1
in volume fractions θ and $(1-\theta)$, as $A_1 \to 0$.

We refer to the elements of this set as "effective moduli of perforated composites with volume fraction θ of material." This is to some extent a misnomer: composites created by the formation of cracks are *not* included in the set defined by (3.13).

In essence, our procedure has the effect of interchanging two limits. Shape optimization in its raw form requires that we set $A_1=0$ first, then optimize. In fact we are optimizing first, then setting $A_1=0$. We claim that for compliance optimization (though not necessarily for other design criteria) this procedure is both physically and mathematically correct.

We give the physical argument first. The main effect of our procedure is to exclude from consideration the formation of cracks. However, for compliance optimization it is *never* advantageous to form a crack. This is most easily seen from the principle of minimum elastic energy, whose minimum value is a negative constant times the compliance:

$$\min_{v} \frac{1}{2} \int_{\Omega \setminus H} \langle A_0 e(v), e(v) \rangle - \int_{\partial\Omega} \langle v, f \rangle = -\frac{1}{2} c[H]. \tag{3.14}$$

Introducing a crack along a surface S permits v to be discontinuous on S; this expands the class of test fields for (3.14), so it increases the compliance. (Our optimal designs do use "rank – one composites" where the stress is uniaxial. These resemble arrays of cracks, but they are not the same thing, because the holes have positive volume fraction.)

This argument is of course special to compliance. One can easily imagine other optimization problems in which the formation of cracks might be advantageous. If, for example, the goal were to minimize the elastic energy in a restricted subset $B \subset \Omega$, it could be desirable to shield the entire region B by making a crack around it.

We turn now to the mathematical justification. It consists of the following three observations:

(3.15) *Every "classical design" is also a design in the relaxed sense.*

Indeed, a classical structure can be viewed as having stress $\sigma \equiv 0$ in the "holes" H, and $A(x) = A_0$ in $\Omega \setminus H$. This corresponds to the choice

$$\begin{cases} \theta(x) = 1, & A(x) = A_0 \quad \text{for } x \in \Omega \setminus H \\ \theta(x) = 0, & A(x) = 0 \quad \text{for } x \in H \end{cases} \tag{3.16}$$

in the relaxed problem. The term $\langle A^{-1} \tau, \tau \rangle$ in (3.12) is singular when $A = 0$; its value must be understood as 0 if $A = 0$, $\tau = 0$ and ∞ if $A = 0$, $\tau \neq 0$. With this convention, we can evaluate the relaxed problem at the design (3.16) and the true elastic stress σ associated to H. The result is

$$\int_{\Omega \setminus H} [\langle A_0^{-1} \sigma, \sigma \rangle + \lambda] = c[H] + \lambda |\Omega \setminus H|,$$

as it should be.

(3.17) *In its variational formulation, the relaxed design problem has at least one solution, at least in* 2 D.

Indeed, the relaxed problem has the form

$$\min_{\substack{\operatorname{div} \tau = 0 \\ \tau . n = f}} \int_{\Omega} F_\lambda(\tau). \tag{3.18}$$

In 2 D (and for isotropic A_0), we shall derive a very explicit formula for $F_\lambda(\tau)$. Our analysis shows (*see* Remark 4.2) that $F_\lambda(\tau)$ is *polyconvex*, *i.e.* it can be expressed as a convex function of τ and det τ. It is also continuous in τ, and it has quadratic growth as $\|\tau\| \to \infty$. It is a standard matter in the Calculus of Variations to show, in these circumstances, the existence of a minimizer (*see* e. g. [K & S, 1986]).

(3.19) *Every optimal relaxed design can be approximated by classical ones.*

Briefly, the reason is that our procedure of homogenizing first, then passing to the degenerate limit, is valid for the microstructures that arise in an optimal design. Let us spell this out. Suppose that τ achieves the minimum in (3.18). By a standard argument from numerical analysis, there is a piecewise polynomial τ^ε which approaches optimal behavior:

$$\operatorname{div} \tau^\varepsilon = 0, \qquad \tau^\varepsilon . n = f, \qquad \int_\Omega |\tau^\varepsilon - \tau|^2 \leqq \varepsilon,$$

$$\int_\Omega F_\lambda(\tau^\varepsilon) \leqq \int_\Omega F_\lambda(\tau) + \varepsilon.$$

Appealing to the calculation in Section 4, there is a "design" corresponding to τ^ε. Where $\tau^\varepsilon(x) = 0$ it has a hole ($\theta^\varepsilon(x) = 0$, $A^\varepsilon(x) = 0$); where $|\tau_1^\varepsilon| + |\tau_2^\varepsilon|$ is larger than a certain constant it is solid ($\theta^\varepsilon(x) = 1$, $A^\varepsilon(x) = A_0$); elsewhere it consists of a sequentially laminated composite of rank one (where det $\tau^\varepsilon = 0$) or rank two (where det $\tau^\varepsilon \neq 0$). By moving a little extra material into the regions where $\tau^\varepsilon = 0$ or det $\tau^\varepsilon = 0$, one obtains a "design" $\tilde{A}^\varepsilon(x)$, $\tilde{\theta}^\varepsilon(x)$ which is still nearly optimal, but which has no holes and uses only composites of rank two. One can even choose $\tilde{A}^\varepsilon(x)$ and $\tilde{\theta}^\varepsilon(x)$ to be piecewise constant (on a mesh much finer than that of τ^ε). Now, the microstructure of a sequentially laminated composite of rank 2 consists of arrays of rectangular holes, properly arranged in a matrix of material A_0. For such a microstructure the effective behavior can indeed be calculated by homogenizing first, then passing to the degenrate limit. (The corresponding statement for periodic microstructures is proved in Cioranescu & Saint Jean Paulin [1979]; for rank-2 laminates one must argue a little differently, using the method of two-scale convergence.) By taking the microstructure to be small but finite in length scale, we obtain a classical design H for which

$$c[H] + \lambda |\Omega \setminus H| \leqq \int_\Omega [\langle (\tilde{A}^\varepsilon)^{-1} \tau^\varepsilon, \tau^\varepsilon \rangle + \lambda \tilde{\theta}^\varepsilon] + \varepsilon$$

$$\leqq \int_\Omega F_\lambda(\tau) + 2\varepsilon.$$

Of course, this design may have a great many holes. Still, it represents a classical design whose behavior is within 2ε of being optimal.

The arguments presented for (3.15) and (3.19) work essentially the same in 3 D as in 2 D. The argument for (3.17) is less satisfactory, since it uses the polyconvexity of $F_\lambda(\tau)$ — which we know only in 2 D, with A_0 isotropic, and only by inspection. It might be possible to prove quasiconvexity by the method of Kohn & Vogelius [1987]; though weaker than polyconvexity, this would be sufficient for the existence of optimal designs in greater generality. Such an argument, however, is not entirely strainghtforward, and we have not done it.

4. Explicit evaluation of the optimal composites in two space dimensions

The goal of this section is to calculate $F_\lambda(\tau)$ explicitly in two space dimensions. We recall the definition:

$$F_\lambda(\tau)=\min_{0\leqq\theta\leqq 1}\ \min_{A\in G_\theta}\left[\langle A^{-1}\tau,\tau\rangle+\lambda\theta\right]. \tag{4.1}$$

We shall first do the optimization over G_θ, leading to an explicit formula for

$$f(\theta,\tau)=\min_{A\in G_\theta}\langle A^{-1}\tau,\tau\rangle. \tag{4.2}$$

Then a one-dimensional optimization over θ will lead to F_λ:

$$F_\lambda(\tau)=\min_{0\leqq\theta\leqq 1}\left[f(\theta,\tau)+\lambda\theta\right]. \tag{4.3}$$

Evaluating (4.2) amounts physically to finding the "most rigid" composite with volume fraction θ, when the (macroscopic) stress is τ. Mathematically, we seek the extremal value of the specific function $A\mapsto\langle A^{-1}\tau,\tau\rangle$ as A ranges over the set G_θ.

It should be emphasized that there is no restriction in (4.2) upon the symmetry of A. Even if the original Hooke's law A_0 is isotropic, as we shall assume, the microstructure can introduce anisotropy.

There are actually two somewhat different methods for evaluating (4.2). The first makes use of what is now called the "translation method" [G & C, 1987]. The second is based on the "Hashin-Shtrikman variational principle" as developed and applied in ([A & K, 1992*a*, *b*]; [A, 1987]; [K & L, 1988]; and [M & K, 1988]). We prefer the latter because it is more systematic.

The analysis of [A & K, 1992*b*] amounts to an algorithm for evaluating (4.2). The answer is simpler to explain than the algorithm, so we present it first. We restrict our attention to two space dimensions. The basic elastic material A_0 is presumed isotropic, with bulk modulus $\kappa>0$ and shear modulus $\mu>0$:

$$A_0\xi=\kappa(\operatorname{tr}\xi)I+2\mu\left(\xi-\frac{1}{2}(\operatorname{tr}\xi)I\right). \tag{4.4}$$

Let τ_1 and τ_2 be the eigenvalues of τ (the principal stresses). Then the optimal value of (4.2) turns out to be

$$f(\theta,\tau)=\langle A_0^{-1}\tau,\tau\rangle+\frac{1-\theta}{\theta}\,\frac{\kappa+\mu}{4\kappa\mu}(|\tau_1|+|\tau_2|)^2. \tag{4.5}$$

It is achieved by a "sequentially laminated microstructure of rank 2", represented schematically in Figure 1. There are two length scales, $\varepsilon_1 \ll \varepsilon_2 \ll 1$. On the longer length scale

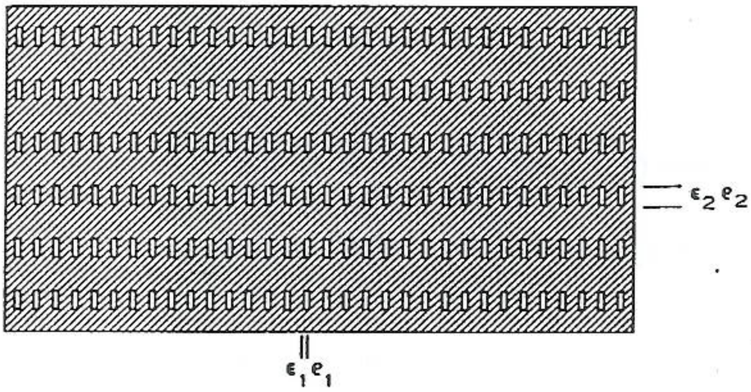


Fig. 1. — Microstructure of a sequentially laminated composite. Long, thin rectangular holes, in a matrix of material A_0.

ε_2, the microstructure is a laminated mixture of A_0 (volume fraction $1-\rho_2$) and a certain composite C (volume fraction ρ_2), in layers orthogonal to e_2 (the eigenvector of τ_2). The shorter length scale is that of C: it is a layered arrangement of A_0 (volume fraction $1-\rho_1$) with void (volume fraction ρ_1), in layers orthogonal to e_1 (the eigenvector of τ_1). The volume fractions ρ_1 and ρ_2 are determined by the formulas

$$1-\rho_1=\theta\,\frac{|\tau_2|}{|\tau_1|+|\tau_2|},\qquad \rho_2=\frac{1-\theta}{\rho_1}. \tag{4.6}$$

Notice that the total volume fraction of material is $(1-\rho_2)+\rho_2(1-\rho_1)=\theta$. As the figure makes clear, this microstructure consists of long, thin rectangular holes of length $\rho_2\,\varepsilon_2$ and width $\rho_1\,\varepsilon_1$, appropriately arranged in a matrix of A_0.

Remark 4.1. — The extremal *value* of (4.2) is uniquely determined by θ and τ. However, the optimal microstructure is *not unique*. We described one example of an optimal microstructure in the preceding paragraph. A second one is obtained by renumbering the principal stresses. When τ is a multiple of the identity the well-known "concentric sphere construction" gives another, totally different, optimal microstructure ([Christensen, 1991]; [Hashin, 1962]). Even the optimal Hooke's law A can be nonunique. For example, if τ is a multiple of the identity then the concentric sphere construction gives an isotropic A, while the second-rank laminate described by (4.6) gives an A which is merely orthotropic. It *is* possible to get an extremal isotropic composite by sequential lamination, but this requires the use of a third-rank laminate (*see* [F & M, 1986]).

Given (4.5), the evaluation of $F_\lambda(\tau)$ is elementary. The definition (4.3) becomes

$$F_\lambda(\tau)=\langle A_0^{-1}\tau,\tau\rangle+\min_{0\le\theta\le1}\left[\frac{1-\theta}{\theta}\,\frac{\kappa+\mu}{4\kappa\mu}\,(|\tau_1|+|\tau_2|)^2+\lambda\theta\right]. \tag{4.7}$$

The optimal θ satisfies

$$\theta=\left(\frac{\kappa+\mu}{4\,k\mu}\right)^{1/2}\lambda^{-1/2}(|\tau_1|+|\tau_2|) \tag{4.8}$$

if this quantity is less than 1, and $\theta=1$ otherwise. If we set

$$\rho_\lambda(\tau)=\left(\frac{\kappa+\mu}{4\,\kappa\mu}\right)^{1/2}\lambda^{-1/2}(|\tau_1|+|\tau_2|), \tag{4.9}$$

then substitution into (4.7) gives

$$F_\lambda(\tau)=\begin{cases}\langle A_0^{-1}\tau,\,\tau\rangle+\lambda & \text{if}\quad \rho_\lambda(\tau)\geqq 1\\ \langle A_0^{-1}\tau,\,\tau\rangle+\lambda\rho_\lambda(2-\rho_\lambda) & \text{if}\quad \rho_\lambda(\tau)\leqq 1.\end{cases} \tag{4.10}$$

This is the desired explicit formula for the integrand of the relaxed variational problem (2.16). To avoid any possible confusion, we note that for the isotropic A_0 (4.4) in two space dimensions,

$$\langle A_0^{-1}\tau,\,\tau\rangle=\frac{\mu-\kappa}{4\,\kappa\mu}(\operatorname{tr}\tau)^2+\frac{1}{2\,\mu}\|\tau\|^2. \tag{4.11}$$

Here $\|\tau\|^2=\tau_{11}^2+2\tau_{12}^2+\tau_{22}^2$, in terms of the components of τ. (Equivalently $\|\tau\|^2=\tau_1^2+\tau_2^2$, in terms of the eigenvalues of τ.)

It is natural that (4.10) should have two distinct regimes. At points where the stress is large, *i.e.* where $\rho_\lambda(\tau(x))\geqq 1$, the optimal design keeps the original material intact. Thus the volume fraction is $\theta(x)=1$ and the Hooke's law is $A(x)=A_0$. At points where the stress is smaller, *i.e.* where $\rho_\lambda(\tau(x))\leqq 1$, the optimal design has a perforated composite. The volume fraction of material is $\theta(x)=\rho_\lambda(\tau(x))$, and the Hooke's law $A(x)$ is extremal for (4.2) with $\theta=\theta(x)$ and $\tau=\tau(x)$. It is possible that $\tau(x)$ might vanish on some subset $H\subset\Omega$. Where the stress is zero no material is required, so $\theta=0$ and H is a "hole".

We discussed in Section 2 the distinction between permitting *some* composites as microstructures, and permitting *all* of them. Since an optimal microstructure can always be found within the class of sequentially laminated ones of rank two, it is obviously sufficient to consider only microstructures of that type. This approach has been explored by Bendsoe, Haber, and Jog [B, 1989], [J *et al.*, 1992]. Since the optimization over microstructures is done numerically rather than analytically in [B, 1989], that work does not obtain explicit formulas such as (4.5) or (4.8).

Remark 4.2. — The relaxed integrand (4.10) happens to be *polyconvex*. This means $F_\lambda(\tau)$ can be expressed as

$$F_\lambda(\tau)=\Phi_\lambda(\tau,\,\det\tau),$$

with Φ_λ a convex function of its five variables. This structural property is important, because it implies that the relaxed problem has at least one solution (*see* Sec. 3). To show that F_λ is polyconvex, we rewrite (4.10)—making use of (4.11)—as

$$F_\lambda(\tau)=\lambda\, G(\tilde{\tau})+2\lambda\,\frac{\mu-\kappa}{\mu+\kappa}\det\tilde{\tau}, \tag{4.12}$$

in which

$$\tilde{\tau}=\left(\frac{\kappa+\mu}{4\kappa\mu}\right)^{1/2}\lambda^{-1/2}\tau$$

and

$$G(\tilde{\tau})=\begin{cases}\|\tilde{\tau}\|^2+1 & \text{if } \tilde{\rho}(\tilde{\tau})\geqq 1\\ 2\tilde{\rho}(\tilde{\tau})-2|\det\tilde{\tau}| & \text{if } \tilde{\rho}(\tilde{\tau})\leqq 1,\end{cases}$$

with the convention

$$\tilde{\rho}(\tilde{\tau})=|\tilde{\tau}_1|+|\tilde{\tau}_2|.$$

The function $G(\tilde{\tau})$ is polyconvex: *see* Lemma 3.4 of [K & S, 1986]. The second term of (4.12) is linear in $\det\tilde{\tau}$, hence certainly polyconvex. So $F_\lambda(\tau)$, being the sum of two polyconvex functions, it itself polyconvex.

The remainder of this section provides the justification of (4.5). Readers who are willing to accept that formula can skip directly to Section 5 without loss of continuity. We shall draw heavily upon our recent exposition [A & K, 1992*b*] concerning optimal bounds on elastic energies.

Our starting point is the following finite-dimensional concave variational principle for $f(\theta,\tau)$:

$$f(\theta,\tau)-\langle A_0^{-1}\tau,\tau\rangle=(1-\theta)\sup_{\eta}\{2\langle\eta,\tau\rangle-\theta g_c(\eta)\}. \tag{4.13}$$

This is formula (6.9) of [A & K, 1992*b*], specialized to the case of a perforated composite. The maximization is over all symmetric matrices η. The function $g_c(\eta)$ is defined by

$$g_c(\eta)=\sup_{|k|=1}|\pi_{A_0^{-1/2}W(k)}A_0^{1/2}\eta|^2, \tag{4.14}$$

in which k ranges over unit vectors $W(k)$ is the space of symmetric matrices ξ satisfying $\xi.k=0$; and π_S denotes orthogonal projection onto the subspace S. Notice that (4.14) gives g_c as a maximum of quadratic functions, so it is a convex function of η. One obtains (4.5) by doing the optimizations in (4.14) and (4.13) explicitly.

The function $g_c(\eta)$ has the formula

$$g_c(\eta)=\frac{4\kappa\mu}{\kappa+\mu}\max\{\eta_1^2,\eta_2^2\}, \tag{4.15}$$

in which η_1 and η_2 are the eigenvalues of η. This is easily deduced from formulas (6.2), (6.10), (6.18), and (7.7) of [A & K, 1992*b*].

Turning to the variational principle for $f(\theta, \tau)$, (4.13), the main point is to evaluate

$$\sup_{\eta} \{2\langle \eta, \tau\rangle - \theta g_c(\eta)\}. \tag{4.16}$$

It is a standard result that η should be simultaneously diagonal with τ. The optimal choice is easily seen to be $\eta_1 = \operatorname{sgn}(\tau_1) t$, $\eta_2 = \operatorname{sgn}(\tau_2) t$, with t achieving

$$\sup_t 2(|\tau_1| + |\tau_2|) t - \frac{4\kappa\mu}{\kappa+\mu} \theta t^2.$$

The best t is

$$t = (|\tau_1| + |\tau_2|) . \frac{\kappa+\mu}{4\kappa\mu\theta}, \tag{4.17}$$

yielding

$$\frac{\kappa+\mu}{4\kappa\mu\theta} (|\tau_1| + |\tau_2|)^2$$

for the maximum of (4.16). Substitution into (4.13) gives

$$f(\theta, \tau) - \langle A_0^{-1} \tau, \tau\rangle = \frac{1-\theta}{\theta} \frac{\kappa+\mu}{4\kappa\mu} (|\tau_1| + |\tau_2|)^2,$$

which is the same as (4.5).

It remains to determine the microstructure of an optimal composite. The theory presented in [A & K, 1992 *b*] gives the microstructure in terms of the optimality condition for (4.13):

$$2\tau \in \theta\, \partial g_c(\eta). \tag{4.18}$$

where $\partial g_c(\eta)$ is the subdifferential of g_c at the optimal η. Following the notation of [A & K, 1992 *b*], we denote by $f_{A_0}^c(k)$ the degenerate Hooke's law associated to the quadratic form

$$\langle f_{A_0}^c(k) \eta, \eta\rangle = |\pi_{A_0^{-1/2} W(k)} A_0^{1/2} \eta|^2. \tag{4.19}$$

Then (4.18) is equivalent to

$$\tau = \theta \sum_{i=1}^{p} m_i f_{A_0}^c(k_i) \eta, \tag{4.20}$$

with $0 \leqq m_i \leqq 1$, $\sum m_i = 1$, and k_i extremal for (4.14) for each i. Arguing as for Theorem 3.5 of [A & K, 1992 *b*] [*see* also formulas (6.11)-(6.12) of that paper], one shows that a composite whose Hooke's law A satisfies

$$(1-\theta)(A^{-1} - A_0^{-1})^{-1} = \theta \sum_{i=1}^{p} m_i f_{A_0}^c(k_i) \tag{4.21}$$

is in fact extremal. Moreover, such a composite can be achieved by sequential lamination. To be specific, consider a sequence $C^{(1)}$, $C^{(2)}$, ... of perforated composites obtained as follows: $C^{(1)}$ is obtained by layering void (volume fraction ρ_1) and A_0 (volume fraction $1-\rho_1$) in layers orthogonal to k_1; and for $r \geqq 1$, $C^{(r)}$ is obtained by layering $C^{(r-1)}$ (volume fraction ρ_r) with A_0 (volume fraction $1-\rho_r$) in layers orthogonal to k_r. If $\rho_1, \ldots, \rho_p$ are chosen so that

$$(4.22) \qquad (1-\rho_r) \prod_{i=1}^{r-1} \rho_i = \theta m_r, \qquad 1 \leqq r \leqq p,$$

then the resulting composite $C^{(p)}$ satisfies (4.21). (This is the analogue for complementary energy of Proposition 3.2 of [A & K, 1992 *b*]; *see* [F & M, 1986] or [K & L, 1988] for more detail.)

We shall show that at the optimal η, (4.12) holds with $p=2$,

$$(4.23) \qquad m_1 = \frac{|\tau_2|}{|\tau_1|+|\tau_2|}, \qquad m_2 = \frac{|\tau_1|}{|\tau_1|+|\tau_2|},$$

and k_i = the eigenvector of τ associated to τ_i. A bit of calculation based on the results in [A & K, 1992 *b*] shows that

$$(4.25) \qquad f^c_{A_0}(k)\eta = A_0\eta - 4\mu[(\eta k)\odot k - \langle \eta k, k\rangle k\odot k] - \frac{1}{\kappa+\mu}[(\kappa-\mu)\operatorname{tr}\eta + 2\mu\langle \eta k, k\rangle][(\kappa-\mu)I + 2\mu k\odot k],$$

where we use the notation $v\odot k = (v\otimes k + k\otimes v)/2$. Consider first the case $\tau_1\tau_2 \leqq 0$. Replacing τ by $-\tau$ if necessary, we may suppose that $\tau_1 \geqq 0 \geqq \tau_2$. Then the optimal η for (4.13) is

$$\eta = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, \qquad t = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\tau_1|+|\tau_2|)$$

[*see* (4.17)]. In this case the only extremal k for the definition of $g(\eta)$ are the eigenvectors of τ. Both eigenvectors qualify, since $\eta_1^2 = \eta_2^2 = t^2$. For these η and k, (4.24) yields

$$\theta f^c_{A_0}(k_1)\eta = -(|\tau_1|+|\tau_2|)k_2\odot k_2$$
$$\theta f^c_{A_0}(k_2)\eta = -(|\tau_1|+|\tau_2|)k_1\odot k_1.$$

So (4.20) reduces to

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = (|\tau_1|+|\tau_2|)\begin{pmatrix} m_2 & 0 \\ 0 & -m_1 \end{pmatrix},$$

which yields (4.23).

The other case is $\tau_1 \tau_2 \geqq 0$. Replacing τ by $-\tau$ if necessary, we may suppose that $\tau_1 \geqq 0$, $\tau_2 \geqq 0$. Then the optimal η for (4.13) is

$$\eta = t.\mathrm{I}, \qquad t = \frac{\kappa+\mu}{4\kappa\mu\theta}(|\tau_1|+|\tau_2|).$$

Since η is isotropic, every k is extremal in the definition of $g(\eta)$. We may nevertheless take k_1 and k_2 to be eigenvectors of τ, as before. With this choice of η and k, (4.24) yields

$$\theta f(k_1)\eta = (|\tau_1|+|\tau_2|) k_2 \odot k_2$$
$$\theta f(k_2)\eta = (|\tau_1|+|\tau_2|) k_1 \odot k_1,$$

and (4.20) reduces to

$$\begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} = (|\tau_1|+|\tau_2|) \begin{pmatrix} m_2 & 0 \\ 0 & m_1 \end{pmatrix}.$$

This once again yields (4.23).

The description of the optimal sequentially laminated composites follows easily from these results. When $p=2$, (4.22) reduces to

$$(1-\rho_1) = \theta m_1, \qquad (1-\rho_2)\rho_1 = \theta m_2.$$

This is readily seen to be the same as (4.6) when m_1, m_2 are given by (4.23).

5. Numerical methods and results

We recapitulate the form of the relaxed problem in 2 D. For any fixed $\lambda > 0$, minimizing *compliance* + λ *area* is equivalent to solving an explicit variational problem

$$\min_{\substack{\operatorname{div}\tau=0 \\ \tau.n=f}} \int_\Omega \mathrm{F}_\lambda(\tau), \tag{5.1}$$

The integrand is given by (4.10). If τ solves (5.1), then the associated design has no holes where $\rho_\lambda(\tau) \geqq 1$; it has no material where $\rho_\lambda(\tau)=0$; and it has a composite of density $\theta = \rho_\lambda(\tau)$ where $0 < \rho_\lambda(\tau) < 1$. The microstructure of the optimal design is not uniquely determined, but if desired it can be chosen as a sequentially laminated composite of rank 2. [At points where τ has rank one the construction degenerates to a laminated composite of rank one, *i.e.* an array of fibers in the direction of principal stress, as is readily seen from (4.23).]

The minimization of (5.1) must naturally be done numerically. Discretizing (5.1) via finite elements presents no particular problem. Unlike most classical optimization codes, we can use a design-independent discretization. In other words, we work with a fixed computational mesh on Ω. There is no need for front-tracking or refinement near the boundary of an optimal shape.

There is one subtlety, however. It is the fact that $F_\lambda(\tau)$ is not a differentiable function of τ. The problem is that

$$\rho_\lambda(\tau) = \text{Const.}\ (|\tau_1| + |\tau_2|)$$

has discontinuous derivative wherever $\tau_1 = 0$ or $\tau_2 = 0$.

This lack of smoothness is not an accident, but an intrinsic feature of the relaxation process. When $\tau_2 = 0$, the optimal composition consists of fibers of material. As τ_2 departs from zero, the optimal microstructure acquires some "struts" orthogonal to the fibers

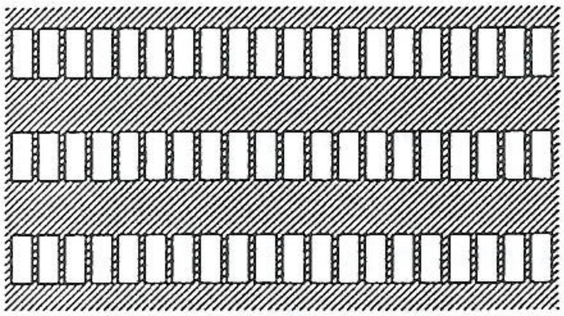


Fig. 2. — The optimal sequentially laminated microstructure when one principal stress is almost zero. The long, thin holes are separated by thin struts in the associated principal direction.

(see *Fig*. 2). The density of the struts is approximately proportional to $|\tau_2|$, by (4.6). Therefore the stress in any single strut is of order 1, and the overall energy of the struts is *density* $\times$ *energy* $\approx c|\tau_2|$. Hence the nondifferentiable character of F_λ near $\tau_2 = 0$.

We note that in the limit as $\tau \to 0$,

$$F_\lambda(\tau) \approx 2\lambda\rho(\tau)$$

has *linear* growth in τ. This is another reflection of the lack of smoothness, and it, too, has a physically intuitive explanation. For any fixed composite the elastic energy is quadratic in τ. In the optimal design, however, the composite itself depends on τ. Since the volume fraction $\theta = \rho_\lambda(\tau)$ is linear in τ, the Hooke's law $A(x)$ is of order $|\tau|$ for τ near zero. Hence the compliance of the optimal design, $\langle A^{-1}\tau, \tau\rangle$, is linear rather than quadratic in τ. Since the density is also linear, so is $F_\lambda(\tau) =$ *compliance* $+ \lambda$. *density*.

It would be interesting and worthwhile to apply techniques from nonsmooth optimization. We did not attempt this; rather, we regularized the singularity. Noticing that

$$(|\tau_1| + |\tau_2|)^2 = \|\tau\|^2 + 2|\det\tau|,$$

we made the approximation

$$(|\tau_1| + |\tau_2|)^2 \approx \|\tau\|^2 + 2(\varepsilon^2 + (\det\tau)^2)^{1/2},$$

with $\varepsilon \ll 1$. Thus our numerical F_λ was still given by (4.10), but using

$$(5.2) \qquad \rho_\lambda(\tau) = \left(\frac{\kappa+\mu}{4\kappa\mu}\right)^{1/2} \lambda^{-1/2} [\|\tau\|^2 + 2(\varepsilon^2 + (\det\tau)^2)^{1/2}]^{1/2}$$

in place of the exact formula (4.9). The effect of this regularization depends on both $\|\tau\|$ and $|\det \tau|$. It is greatest, of order $\sqrt{\varepsilon}$, when $\tau=0$. It is also significant, of order ε, when $\|\tau\|\neq 0$ but $\det \tau=0$. It is smallest, of order ε^2, when $\det \tau\neq 0$.

Our choice (5.2) is by no means the only way of regularizing the problem. A physically natural alternative would be to treat the "holes" as being occupied by a nearly degenerate elastic material. The method of Section 4 can be used to determine an associated relaxed problem $F_\lambda^{reg}(\tau)$. The formula for F_λ^{reg} is more complicated than that for F_λ, because there are several regimes rather than just two. One can show that $F_\lambda^{reg}(\tau)$ is a continuously differentiable function of τ, by arguing as in Remark 3.8 of [A & K, 1992*b*]. The qualitative bechavior of $F_\lambda^{reg}(\tau)$ is quite different from that of (5.2). The distinction in something like that between

$$f^1(x)=(x^2+\varepsilon^2)^{1/2}$$

and

$$f_2(x)=\begin{cases} \dfrac{\varepsilon}{2}+\dfrac{1}{2\varepsilon}x^2, & x\leqq\varepsilon \\ x, & x\geqq\varepsilon, \end{cases}$$

two very different regularizations of $|x|$.

Now we discuss how (5.1) was discretized. On a simply connected domain Ω, every solution of $\operatorname{div}\tau=0$ comes from an Airy stress function ψ:

$$\tau=\begin{pmatrix} \psi_{22} & -\psi_{12} \\ -\psi_{12} & \psi_{11} \end{pmatrix}, \tag{5.3}$$

where $\psi_{ij}=\partial^2\psi/\partial x_i\,\partial x_j$. The traction boundary condition $\tau.n=f$ determines ψ and $\nabla\psi$ at $\partial\Omega$. With this substitution (5.1) takes the form

$$\min_{\psi-\psi_0\in H_0^2(\Omega)}\int_\Omega \tilde{F}_\lambda(\nabla\nabla\psi), \tag{5.4}$$

analogous to a problem of plate theory. We calculated the boundary condition ψ_0 for (5.4) exactly, by hand, though of course this could have been done numerically instead.

Notice that when $\lambda=0$, $F_\lambda(\tau)=\langle A_0^{-1}\tau, \tau\rangle$, and (5.1) is just the principle of minimum complementary energy for the original design (with no holes). Dual methods for solving elasticity problems have received considerable attention, e. g. [Fraejis de Veubeke, 1965]. Our approach to (5.1) is similar—except for the fact that $F_\lambda(\tau)$ is not quadratic in τ when $\lambda\neq 0$.

Our goal was to demonstrate the feasibility of the relaxed approach, not to develop a general-purpose structural optimization code. We therefore worked exclusively with rectangular domains Ω. The variational problem (5.4) was discretized using the Clough-Tocher elements [Ciarlet, 1974], which are piecewise cubic and C^1. This is perhaps the simplest choice of conforming finite element for a problem of plate theory. (It is completely equivalent to discretizing (5.1) directly using a mixed method, with piecewise linear "stress" and piecewise constant "strain" [Johnson & Mercier, 1978].) The rectangular domain Ω is divided into congruent right triangles as shown in Figure 3. The degrees

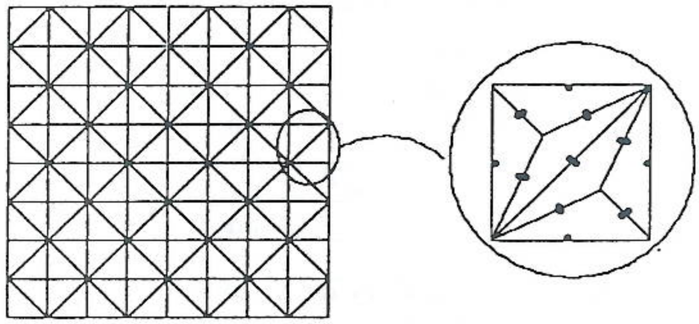


Fig. 3. – Our triangularization of a square (8 × 8 mesh). Each right triangle is subdivised into three substriangles (inset). The Clough-Tocher elements are cubic on each subtriangle The distinguished points marked in the inset are our interpolation points.

of freedom for the Clough-Tocher elements are the nodal values of ψ, $\partial\psi/\partial x_1$, and $\partial\psi/\partial x_2$, along with the normal derivative $\partial\psi/\partial n$ at the midpoint of each line segment. Each right triangle is divided into three subtriangles which meet at the centroid. The function ψ is a cubic polynomial in each subtriangle, C^1 across subtriangle boundaries. We approximated the integral over any subtriangle by a 3-point interpolation rule, using the midpoints of the sides as interpolation points. Notice that τ is linear on each subtriangle; our interpolation rule would therefore be exact if $F_\lambda(\tau)$ were a quadratic function of τ.

The discretized version of (5.4) is a large, nonlinear minimization problem for the degrees of freedom that determine ψ. We used the conjugate gradient method (with no preconditioner) to find the minimum. The efficiency of the method depends greatly on the choice of ε in (5.2). When ε is small, say $\varepsilon = 10^{-4}$, the optimal step size becomes very small, so that many conjugate gradient steps are required. When ε is larger, say $\varepsilon = 10^{-2}$, the method is much more efficient. The effect of the regularization is perceptible, however the qualitative character of the design is not affected (we shall return to this below). The difficulty in handling $\varepsilon \to 0$ is not special to the use of conjugate gradient: any method designed for minimizing C^1 functions would behave similarly.

Most of our calculations were done on a square using an 8×8 or 16×16 mesh. This is not as coarse as it may sound, because of the subtriangles and the piecewise cubic character of ψ. When the mesh is 8×8, ψ is cubic on each of $64 \times 6 = 384$ triangles, and there are 323 degrees of freedom (not counting those which are fixed by the boundary data). When the mesh is 16×16, ψ is cubic on each of $256 \times 6 = 1{,}536$ triangles, and there are 1,411 degrees of freedom.

The numerical optimum ψ determines a piecewise linear τ which is admissible for (5.1), so it gives an upper bound for the true value of the optimum. There is an associated composite, whose microstructure can be read off from τ as described in Section 4. Notice that since τ is not locally constant, neither is the microstructure. It is not easy to represent a stress field graphically, so we shall present here only the volume fraction of material, $\rho_\lambda(\tau)$. The output of our program yields its values at the various interpolation points. This determines a unique piecewise linear function. We sampled it at regularly spaced points, then used a standard graphics package to obtain our figures.

We chose the elastic moduli to be $\kappa=1.0$ and $\mu=0.5$. As λ varies, however, this represents no loss of generality. Indeed, for traction boundary conditions the solution of the relaxed problem depends only on the *single* parameter $(\mu^{-1}+\kappa^{-1})/\lambda$. This is most easily seen from the formula (4.12). The second term is a *null-Lagrangian*; its value

$$2\lambda.\frac{\mu-\kappa}{\mu+\kappa}\int_\Omega \det\tilde{\tau}=\frac{\mu-\kappa}{2\kappa\mu}\int_\Omega \det\tau$$
$$=\frac{\mu-\kappa}{2\kappa\mu}\int_\Omega (\psi_{22}\psi_{11}-\psi_{12}^2)$$

is completely determined by the boundary data. So the optimal design could just as well be obtained by minimizing just the first term of (4.12), which by inspection depends only on $(\mu^{-1}+\kappa^{-1})/\lambda$. [The preceding argument applies *only* for traction boundary conditions at $\partial\Omega$. Our approach can also be used for displacement boundary conditions, and in that case the second term in (4.12) becomes important.] As λ increases with κ and μ held fixed, the compliance of the optimal design increases and the weight decreases. This is immediately evident from the original, unrelaxed formulation of the design problem (2.5).

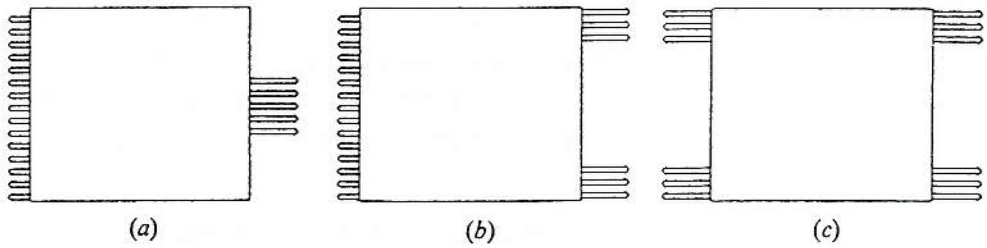


Fig. 4. – Three different load configurations. (*a*) Uniform tension along the left-hand side and the middle portion of the right-hand side; (*b*) Uniform tension along the left-hand side and the upper and lower portions of the right-hand side; (*c*) Uniform tension along the upper and lower parts of opposite sides.

We now present results for the three different loading configurations shown in Figure 4. Figures 5-7 show the volume fraction of material $\rho_\lambda(\tau)$ in some optimal designs. Each figure corresponds to a different load configuration, and to a specific choice of λ. In each case the regularization parameter is $\varepsilon=10^{-2}$ and the numerical grid is 8×8. Notice that these designs make considerable use of composite materials. They also have some "holes," *i.e.* regions where $\rho_\lambda(\tau)\approx 0$. The density can vary rather sharply near the edge of a "hole" (e.g. in the corners of *Fig.* 5). However, it is relatively smooth in the interior of the region where composites occur. Notice that in Figure 7, the design has essentially broken apart into two disjoint substructures. This is of course expected. The vertical variation of $\rho_\lambda(\tau)$ within each substructure is presumably due to the smoothing parameter ε and/or the spatial discretization.

Figure 8 shows how the optimal design changes as λ varies. The different choices of λ lead to optimal designs with different total weights. These examples use the same loads, smoothing parameter, and grid as Figure 5; only the value of λ is being changed. When

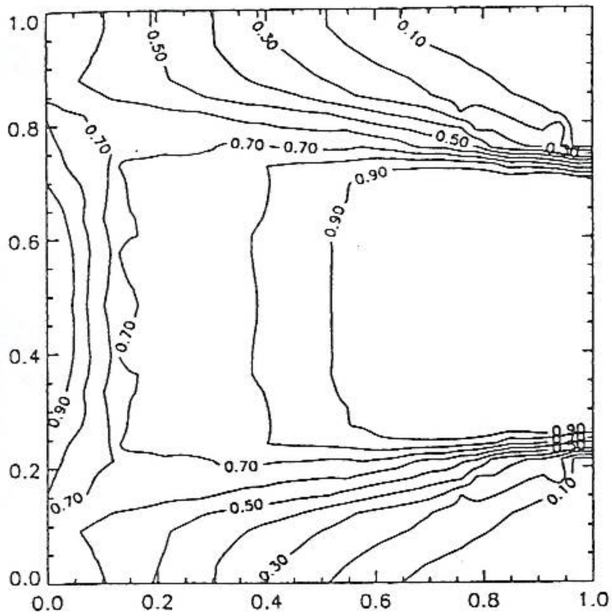


Fig. 5

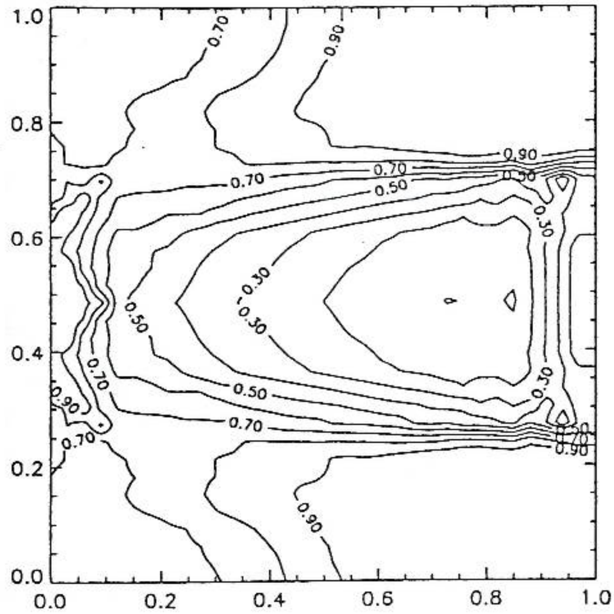


Fig. 6

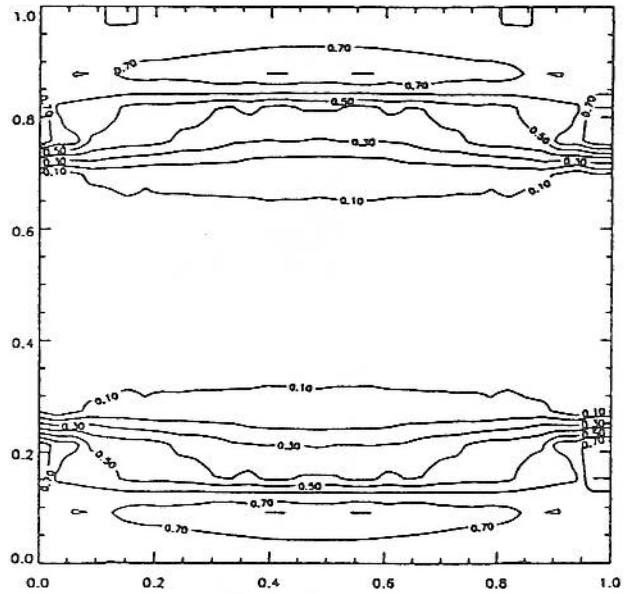


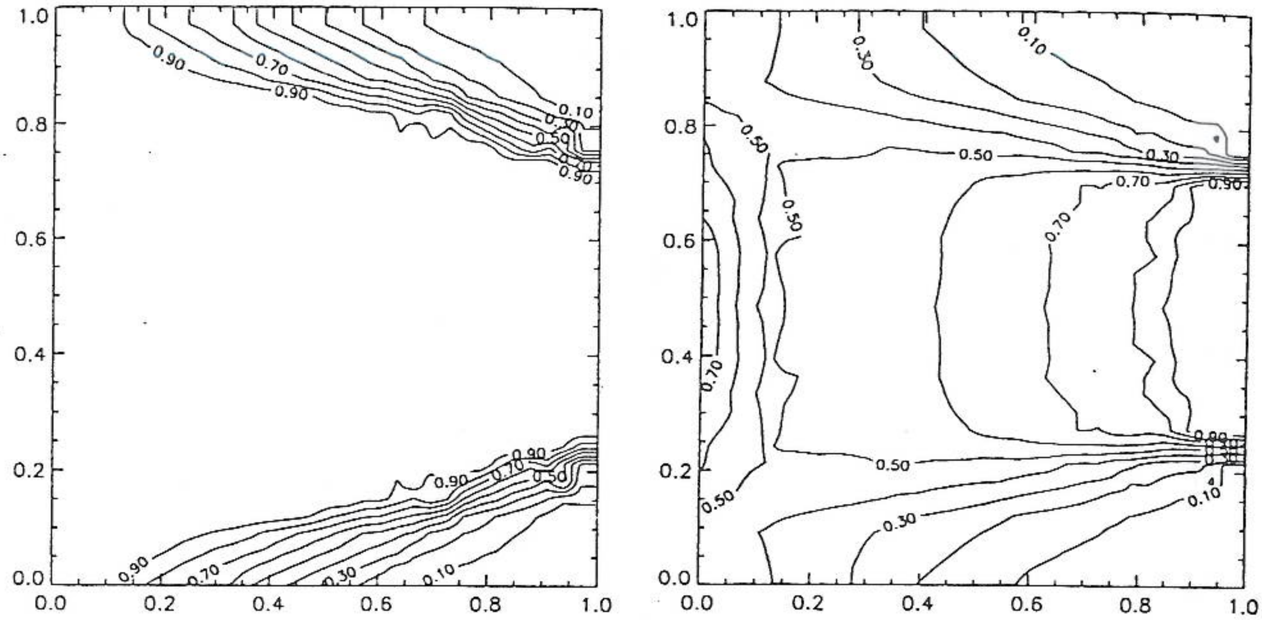
Fig. 7

Fig. 5. — The volume fraction $\rho_\lambda(\tau)$ in the optimal design. The load configuration is given by Figure 4 *a*. The value of λ is 0.5. Approximately 35% of the material has been removed.

Fig. 6. — The volume fraction $\rho_\lambda(\tau)$ in the optimal design. The load configuration is given by Figure 4 *b*. The value of λ is 0.5. Approximately 35% of the material has been removed.

Fig. 7. — The volume fraction $\rho_\lambda(\tau)$ in the optimal design. The load configuration is given by Figure 4 *c*. The value of λ is 2.0

λ is small (*Fig.* 8), the main effect of optimization is to remove the upper and lower right-hand corners. The computed optimum makes some use of composites, but this might be due solely to the effects of discretization and regularization. Our results are thus consistent with the idea that the optimal design should be a classical one when only



(a) (b)

Fig. 8. — Same as Figure 5, except for different choices of λ: (a) $\lambda = 0.2$, approximately 15% of the material is removed; (b) $\lambda = 1.0$, approximately 52% of the material is removed.

a small amount of material is being removed. At the other extreme, when λ is large, rather little material is left. The design has "holes" ($\rho_\lambda(\tau) \approx 0$) in the corners, and appropriately chosen composite materials ($0 < \rho_\lambda(\tau) < 1$) elsewhere.

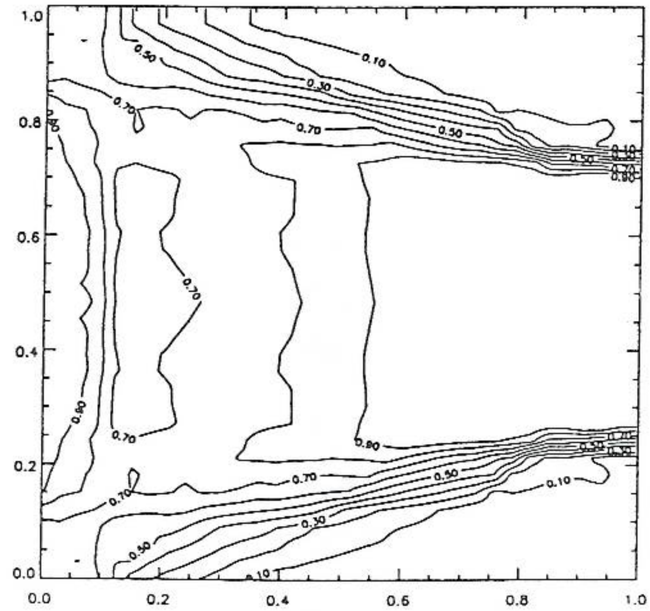


Fig. 9. — Same as Figure 5, except that $\varepsilon = 10^{-4}$ rather than 10^{-2}.

Figure 9 shows the effect of decreasing ε. It uses the same loads, grid, and choice of λ as Figure 5, but takes the smoothing paramater to be $\varepsilon = 10^{-4}$ rather than 10^{-2}. The density gradient is sharper near the corners, and variations due to grid effects are more evident, but the result is qualitatively similar.

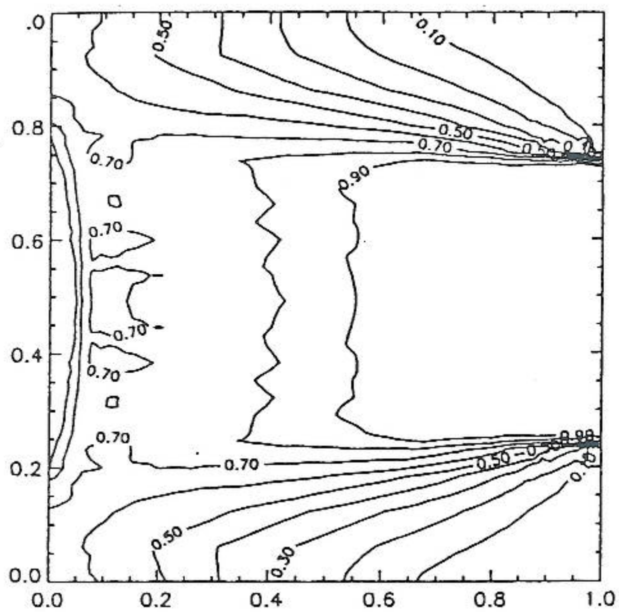


Fig. 10. — Same as Figure 5, except for the use of a 16×16 mesh rather than 8×8.

Figure 10 shows the effect of refining the spatial mesh. It corresponds to Figure 5 except for the use of a 16×16 mesh instead of 8×8. The general features of the design do not change much, though grid effects are somewhat more evident.

Figure 11 shows an example involving a boundary condition other than pure traction. The structure is a rectangle, clamped on the left and loaded in shear along the center of the right hand edge. Arguing as in Section 2, one sees that the associated relaxed design problem is

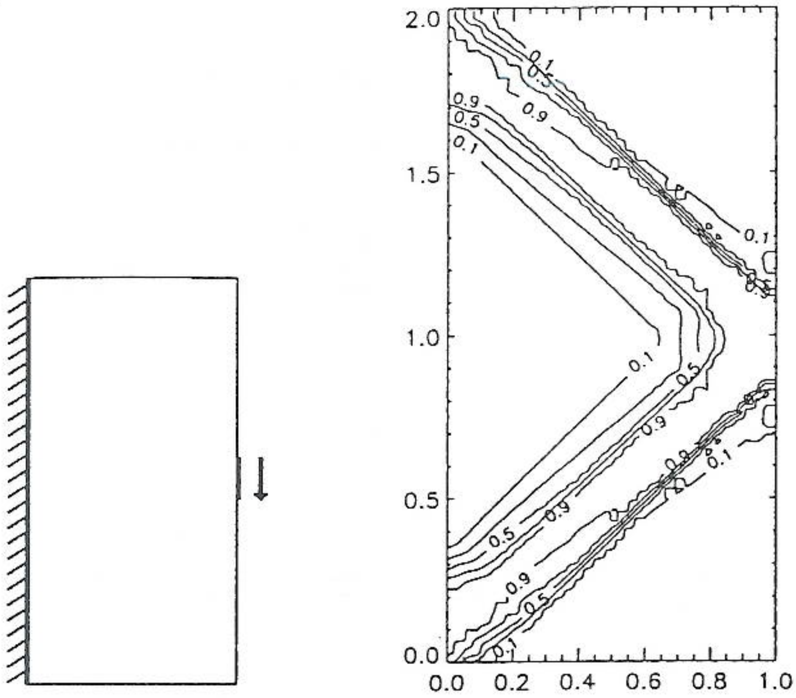
$$\min_{\substack{\operatorname{div} \tau = 0 \\ \tau . n = f \text{ on } \Gamma_0}} \int_{\partial\Omega} F_\lambda(\tau),$$

where $F_\lambda(\tau)$ determined by (4.10), and Γ_0 the portion of $\partial\Omega$ which is not clamped. Notice that in this case our optimal design makes essentially no use of composites.

It would be natural to compare the performance of these optimal designs with ones obtained by a more conventional code. We did not have a conventional structural optimization code at our disposal, so we were unable to do this.

6. The link to Michell trusses

The idea of doing structural optimization in a class of "generalized designs" is not really new. It was first proposed by Michell in 1904, in a striking piece of work that was far ahead of its time. The theory of Michell trusses (for elasticity) and grillage-like continua (for plate theory) has since been developed by many authors, including ([Hemp, 1973]; [Lagache, 1981]; [Prager & Roznavy, 1977], and [Zhou & Rozvany, 1991]). Convenient reviews will be found in ([Kirsch, 1989]; [Rozvany, 1989] and [Rozvany, 1992]). Our



(a) (b)

Fig. 11. – An example with displacement and traction boundary conditions. (a) schematic of loading: left boundary is clamped, right boundary is loaded in the middle; (b) optimal design obtained using $\kappa = 1.0$, $\mu = 0.5$, $\lambda = 0.5$, $\varepsilon = 10^{-4}$, and an 8×16 mesh. Approximately 70% of the material is removed.

variational formulation (5.1) of the relaxed design problem provides a direct link with the theory of Michell trusses. A similar connection has been noted in [R *et al.*, 1987].

To explain, we begin by briefly summarizing the theory of Michell trusses. These structures are made up entirely of linear truss elements, each capable of withstanding a certain tensile or compressive stress. (The possibility of element buckling is ignored.) For a given statically admissible stress field τ, the truss elements should be oriented with the directions of principal stress. The density of truss required to withstand τ is therefore proportional to $|\tau_1| + |\tau_2|$, where τ_1 and τ_2 are the principal stresses. If the truss is confined to a region Ω, then its weight is a constant times $\int_\Omega |\tau_1| + |\tau_2|$. The optimal Michell truss is obtained by minimizing the weight as τ varies among statically admissible stress fields:

$$\min_{\substack{\operatorname{div} \tau = 0 \\ \tau . n = f}} \int_\Omega |\tau_1| + |\tau_2|. \tag{6.1}$$

Because (6.1) is a convex optimization, one can use optimality conditions and convex duality to explore the minimizers. There is a rich theory of explicit solutions. The theory is not yet complete mathematically. Because $|\tau_1| + |\tau_2|$ has linear growth, an optimal Michell truss can be singular, *i.e.* the stress can concentrate on a lower-dimensional set. Also, the boundary condition "$\tau . n = f$ at $\partial\Omega$" may be achieved only in some generalized sense. The situation is similar to that of plastic limit analysis, *see* e.g. ([Temam, 1983]; [Suquet & Bouchitte, 1991]).

Now consider our structural optimization problem (5.1). It corresponds to minimizing *compliance* + λ.*area*, so as $\lambda \to \infty$ the area of the optimal design should tend to zero. One might expect the limiting design to be Michell truss. This is correct for plane stress, as we now show (formally), though it is not correct in three space dimensions.

Let $\tau = \tau^\lambda$ achieve the optimum for given λ. Since div $\tau^\lambda = 0$ and $\tau^\lambda . n = f$ for every λ, it seems reasonable to suppose that τ^λ remains bounded. (Strictly speaking this cannot always be the case, since Michell trusses can have singular elements. For what follows, however, all we really need is $|\tau_1^\lambda| + |\tau_2^\lambda| \ll \sqrt{\lambda}$.) Recall that

$$\rho_\lambda(\tau) = \left(\frac{\kappa+\mu}{4\kappa\mu}\right)^{1/2} \lambda^{-1/2} (|\tau_1| + |\tau_2|). \tag{6.2}$$

We expect that $\rho_\lambda(\tau^\lambda) \ll 1$ as $\lambda \to \infty$, so only the second regime of $F_\lambda(\tau)$ is relevant, and (4.10) becomes

$$F_\lambda(\tau) = \langle A_0^{-1} \tau, \tau \rangle + \lambda \rho_\lambda(\tau)(2 - \rho_\lambda(\tau)). \tag{6.3}$$

Substitution of (6.2) yields

$$F_\lambda(\tau) = c_1 \lambda^{1/2} (|\tau_1| + |\tau_2|) - c_2 (|\tau_1| + |\tau_2|)^2 + \langle A_0^{-1} \tau, \tau \rangle \tag{6.4}$$

with $c_1 = [(\kappa+\mu)/\kappa\mu]^{1/2}$, $c_2 = (\kappa+\mu)/4\kappa\mu$. The first term is the most significant as $\lambda \to \infty$, and it differs from (6.1) only by a scalar factor. Hence our structural optimization problem is asymptotically equivalent to the Michell truss problem (6.1) as $\lambda \to \infty$.

The situation is different in three space dimensions. We compute the relaxed problem below, in Section 7, when the basic material A_0 has Poisson's ratio equal to zero (*see* (7.5), (7.6), and (7.13)]. Repeating in this context the argument given above, one finds that the asymptotic variational problem is

$$\min_{\substack{\operatorname{div} \tau = 0 \\ \tau . n = f}} \int_\Omega h(\tau). \tag{6.5}$$

with

$$h(\tau) = \begin{cases} \sqrt{2}\,[(|\tau_1| + |\tau_2|)^2 + \tau_3^2]^{1/2} & \text{if } |\tau_1| + |\tau_2| \leqq |\tau_3| \\ (|\tau_1| + |\tau_2| + |\tau_3|) & \text{if } |\tau_1| + |\tau_2| \geqq |\tau_3|, \end{cases} \tag{6.6}$$

where $|\tau_1| \leqq |\tau_2| \leqq |\tau_3|$ are the principal stresses (the eigenvalues of τ). The second regime of (6.6) corresponds to a three dimensional Michell truss. The first regime corresponds instead to a sort of honey-comb microstructure, consisting of thin walls appropriately arranged in space. In that regime, such a structure is preferred over a truss.

Returning to the case of plane stress, we note a further connection between our relaxed problem and Michell truss theory. By algebraic manipulation of (6.4), one verifies that in the relaxed regime (where $\rho_\lambda(\tau)<1$), our integrand is

$$F_\lambda(\tau)=c_1\lambda^{1/2}(|\tau_1|+|\tau_2|)+c_2'\tau_1\tau_2-c_2''|\tau_1\tau_2| \tag{6.7}$$

with $c_2'=(\mu-\kappa)/2\kappa\mu$, $c_2''=(\mu+\kappa)/2\kappa\mu$. The first term is that of the Michell truss problem. The second term is a null-Lagrangian, *i.e.* its integral is entirely determined by the boundary values of $\tau.n$. The third term is not a null-Lagrangian, because of the absolute value, but its Euler equation is still an identity in any open set where $\tau_1\tau_2\neq 0$. Thus, for *any* λ, the stresses in an optimal design should resemble those of a Michell truss *wherever* rank-two composites are used. A similarity between relaxed optimal designs and Michell trusses has indeed been observed in some numerical examples, *see* e.g. [S & K, 1991], even when the volume fraction of material is not particularly small.

All these arguments have been concerned with the relaxed variational problem, obtained by permitting extremal composites as structural components. There is one more link to the theory of Michell trusses, this time at the level of microstructure. We have already observed that optimal microstructures are not unique. Our second-rank laminates do not look much like trusses. However, there is another class of optimal microstructures, discovered by S. Vigdergauz [1986] (see also [Grabovsky, in preparation]) for the case det $\tau>0$. They consist of a *periodic* arrangement of properly shaped holes centered at the points of a rectangular lattice. The result is a more or less truss-like structure, with carefully designed joints where the truss members cross one another. If we imagine these microstructures instead of rank-two laminates (which we may do, since both classes achieve the optimal elastic energy bounds), then our optimal designs amount quite literally to large-volume-fraction analogues of Michell trusses where det $\tau>0$.

7. Three space dimensions

Our approach can also be used, at least in principle, for three-dimensional shape optimization. This section discusses the general form of the relaxed problem. We make things completely explicit for the special case when the basic elastic material has Poisson's ratio zero. The case of nonzero Poisson's ratio can also be treated by the same method but the result is messier. As in Section 4, the essense of the matter is an optimal lower bound on $\langle A^{-1}\tau,\tau\rangle$ as A ranges over all perforated composites with volume fraction θ. Such optimal energy bounds for 3D elasticity have also been obtained by Gibiansky & Cherkaev [1987].

We adopt the notation of Sections 2-4. Since we are in three space dimensions, the Hooke's law of the basic elastic material is

$$A_0\xi=\kappa(\operatorname{tr}\xi)I+2\mu\left(\xi-\frac{1}{3}(\operatorname{tr}\xi)I\right)$$

rather than (4.4). The relaxed design problem is still

$$\min_{\substack{\operatorname{div}\tau=0\\ \tau.n=f}}\int_\Omega F_\lambda(\tau),$$

but now $\Omega \subset \mathbf{R}^3$, and the values of τ are 3×3 symmetric matrices. The integrand is still

$$F_\lambda(\tau) = \min_{0 \leqq \theta \leqq 1} [f(\theta, \tau) + \lambda\theta] \tag{7.1}$$

with

$$f(\theta, \tau) = \min_{A \in G_\theta} \langle A^{-1}\tau, \tau \rangle.$$

Our goal is to understand the structure of $F_\lambda(\tau)$.

The Hashin-Shtrikman variational principle is not dimension dependent. Hence the characterization

$$f(\theta, \tau) = \langle A_0^{-1}\tau, \tau \rangle + (1-\theta) \sup_\eta \{2\langle \tau, \eta \rangle - \theta g_c(\eta)\}$$

remains valid in 3 D (subject to the interpretation explained in *Sec.* 3). The definition of g_c is still as before,

$$g_c(\eta) = \sup_{|k|=1} |\pi_{A_0^{-1/2} W(k)} A_0^{1/2} \eta|^2. \tag{7.2}$$

The explicit formula in three space dimensions is easily deduced from formulas in A & K [1992 *b*]:

$$g_c(\eta) = \langle A_0 \eta, \eta \rangle - \frac{\mu}{\gamma+2} \min_{i=1,2,3} (\gamma \operatorname{tr} \eta + 2\eta_i)^2, \tag{7.3}$$

in which we have set

$$\gamma = \frac{3\kappa - 2\mu}{3\mu}$$

We claim that $F_\lambda(\tau)$ *always* has the form

$$F_\lambda(\tau) = \begin{cases} \langle A_0^{-1}\tau, \tau \rangle + \lambda, & \rho_\lambda \geqq 1 \\ \langle A_0^{-1}\tau, \tau \rangle + \lambda\rho_\lambda(2-\rho_\lambda), & \rho_\lambda \leqq 1, \end{cases} \tag{7.4}$$

with $\rho_\lambda = \rho_\lambda(\tau)$ the volume fraction of material, expressed as a function of the stress τ. The function ρ_λ is determined by the relation

$$\rho_\lambda(\tau) = 2\lambda^{-1/2}[g_c^*(\tau)]^{1/2}. \tag{7.5}$$

Here g_c^* is the convex dual of g_c, defined as usual by

$$g_c^*(\tau) = \sup_\eta \{\langle \tau, \eta \rangle - g_c(\eta)\}. \tag{7.6}$$

The proof of (7.4)-(7.5) does not make use of the explicit formula (7.3) for g_c; it depends only on the fact that g_c is convex and homogeneous of degree 2, *i.e.*

$$g_c(t\eta) = t^2 g_c(\eta), \qquad t \in \mathbf{R}. \tag{7.7}$$

These properties are immediate from the definition (7.2). It follows from (7.7) that

$$\sup_{\eta}\{2\langle\tau,\eta\rangle-\theta g_c(\eta)\}=\frac{4}{\theta}g_c^*(\tau) \tag{7.8}$$

Therefore (7.1) becomes

$$F_\lambda(\tau)=\langle A_0^{-1}\tau,\tau\rangle+\min_{0\leqq\theta\leqq 1}\left[\frac{4(1-\theta)}{\theta}g_c^*(\tau)+\lambda\theta\right]. \tag{7.9}$$

The optimal θ satisfies

$$\theta=(4\lambda^{-1}g_c^*(\tau))^{1/2} \tag{7.10}$$

if this quantity is less than one, and $\theta=1$ otherwise. Notice that the right-hand side of (7.10) is precisely $\rho_\lambda(\tau)$. Substitution into (7.9) yields the asserted formula (7.4).

To make the relaxation fully explicit one must calculate $g_c^*(\eta)$. The general case is messy. However the special case $\gamma=0$ is both simple and enlightening. It corresponds to a basic elastic material with Poisson's ratio equal to zero, *i.e.* $\kappa=2\mu/3$ and $A_0\xi=2\mu\xi$. The formula (7.3) becomes

$$g_c(\eta)=2\mu(|\eta|^2-\eta_{\min}^2) \tag{7.11}$$

with $\eta_{\min}^2=\min\{\eta_i^2\}$, and (7.6) becomes

$$g_c^*(\tau)=\sup_{\eta}\{\langle\tau,\eta\rangle-2\mu(|\eta|^2-\eta_{\min}^2)\}. \tag{7.12}$$

The optimal η for (7.12) is simultaneously diagonal with τ. Moreover, if we take the convention $|\tau_1|\leqq|\tau_2|\leqq|\tau_3|$ then the corresponding eigenvalues of η should be ordered similarly, with $\operatorname{sgn}(\eta_i)=\operatorname{sgn}(\tau_i)$. So (7.12) becomes

$$\sup_{|\eta_1|\leqq|\eta_2|\leqq|\eta_3|}\sum|\tau_i|\,|\eta_i|-2\mu(\eta_2^2+\eta_3^2)\}.$$

Clearly the optimal choice has $|\eta_1|=|\eta_2|$. If $|\eta_2|<|\eta_3|$ then the optimality condition is

$$|\tau_1|+|\tau_2|=4\mu|\eta_2|,\qquad |\tau_3|=4\mu|\eta_3|.$$

This is consistent if $|\tau_1|+|\tau_2|<|\tau_3|$. Otherwise the optimal choice has

$$|\eta_1|=|\eta_2|=|\eta_3|=t$$

with

$$|\tau_1|+|\tau_2|+|\tau_3|=8\,\mu t.$$

We thus deduce that

$$g_c^*(\tau)=\begin{cases}\dfrac{1}{8\mu}[(|\tau_1|+|\tau_2|)^2+\tau_3^2] & \text{if } |\tau_1|+|\tau_2|\leqq|\tau_3|\\[2mm] \dfrac{1}{16\mu}(|\tau_1|+|\tau_2|+|\tau_3|)^2 & \text{if } |\tau_1|+|\tau_2|\geqq|\tau_3|.\end{cases} \tag{7.13}$$

Together with (7.4)-(7.5), this makes the relaxed integrand completely explicit when the basic elastic material has Poisson's ratio zero.

In the general case, the formula for $g_c^*(\tau)$ is messier, because there are more distinct regimes. However, it is easy to formulate an algorithm for evaluating $g_c^*(\tau)$ *numerically*.

8. Discussion

This section discusses how our work is related to other recent activity.

8 A. Relation to Bendsoe/Kikuchi/Suzuki

The approach of the recent papers ([B & K, 1988]; [B, 1989]; [S & K, 1991]) is very similar to ours. The main differences are as follows:

(1) for the most part, these papers use suboptimal composites obtained by perforation with a microscopically periodic array of rectangular holes,

(2) the character of the composite is piecewise constant, and the elasticity problem is discretized by a standard strain formulation rather than a dual variational principle,

(3) the optimization is done using the method of optimality criteria rather than a direct method.

Their results are in many respects similar to ours. These seems to be an important difference, however. Their designs make relatively little use of composites, even when a lot of material is removed. Instead, they tend to develop a Michell-truss-like structure on an intermediate length scale, larger than the computational mesh but smaller than the length scale of the domain itself. This feature is especially clear from the examples in [S & K, 1991]. Our examples, by contrast, seem to favor the use of composites. For large λ, the Michell truss limit, our solutions are relatively smooth. They correspond to Michell trusses as *subgrid structure* rather than ones which can be observed macroscopically.

We can only guess at the source of this discrepancy, following a suggestion of A. Cherkaev. Relaxation based on the use of optimal composites led to an integrand $F_\lambda(\tau)$ which is *polyconvex* (*see* Remark 4.2). The associated variational problem is mathematically well behaved, *i.e.* it has a solution. If we had permitted only suboptimal composites, we would have reached a different, slightly larger integrand. It would be almost quasiconvex, but not quite. So its minimization might reasonably lead to spatially oscillatory solutions.

The situation is most easily visualized by considering a scalar optimal design problem (torsional rigidity, electrical resistivity, etc.) rather than elasticity, since quasiconvexity is then replaced by convexity. According to [K & S, 1986], the optimal design problem with no use of composites corresponds to minimizing $\int_\Omega W_0(\tau)$, with

$$W_0(\tau)=\begin{cases}|\tau|^2+\lambda & \tau\neq 0\\ 0, & \tau=0.\end{cases}$$

The use of optimal composites corresponds to minimizing $\int_\Omega W_1(\tau)$, with

$$W_1(\tau)=\begin{cases}|\tau|^2+\lambda, & \lambda^{-1/2}|\tau|\geqq 1\\ 2\sqrt{\lambda}|\tau|, & \lambda^{-1/2}|\tau|\leqq 1,\end{cases}$$

the convexification of W_0. Our regularized relaxed problem corresponds to minimizing $\int_\Omega W_2(\tau)$, with

$$W_2(\tau)=\begin{cases}|\tau|^2+\lambda & \lambda^{-1/2}(|\tau|^2+\varepsilon^2)^{1/2}\geqq 1\\ 2\lambda^{1/2}(|\tau|^2+\varepsilon^2)^{1/2}-\varepsilon^2, & \lambda^{-1/2}(|\tau|^2+\varepsilon^2)^{1/2}\leqq 1,\end{cases}$$

a strictly convex approximation of W_1. The method of Bendsoe-Kikuchi-Suzuki corresponds to minimizing $\int_\Omega W_3(\tau)$ instead, with W_3 a not-quite-convex approximation of W_1. It seems quite plausible that the numerical behavior of these two approximations to the relaxed problem (W_2 *vs.* W_3) might be different.

8 B. Relation to Cherkaev-Gibiansky

The paper [G & C, 1984] executes the analogue of the program developed here, in the context of plate theory. The use of homogenization in conjunction with plate theory has always been a bit controversial. It is perfectly valid, however, provided that the plate is very thin so that its maximum thickness ε and the length scale of thickness variation δ are related by $\varepsilon\ll\delta\ll 1$ [Kohn & Vogelius, 1984]. (The treatment in [Ong *et al.*, 1988] and [R *et al.*, 1987] is different, largely because it corresponds to the case $\delta\ll\varepsilon\ll 1$.) The plates in [G & C, 1984] can have two possible thicknesses, $h_1<h_2$. The case of shape optimization, the focus of our attention here, corresponds to taking $h_1=0$.

The paper [G & C, 1987] addresses the same elasticity problem we consider here. It derives the optimal bound on complementary energy, what we have called $f(\theta,\tau)$. It also considers designs made from two nondegenerate materials, and the corresponding problem in three space dimensions. However, it does not have numerical examples.

The procedure used by Gibiansky and Cherkaev for computing the relaxed problem is somewhat different from ours. Those papers use the "translation method" to derive the lower bound for $\langle A^{-1}\tau,\tau\rangle$ when $A\in G_\theta$. There is no known abstract reason why this bound should be optimal. However, it turns out to be so in each case, by an explicit calculation involving sequential lamination. We prefer our approach, based on the Hashin-Shtrikman variational principle, because its success is guaranteed: the bound obtained this way is known as a matter of theory to be obtimal. The translation method has advantages of its own, however; for example, it is not restricted to mixtures of well-ordered elastic materials.

8 C. Relation to Kohn-Strang

Our variational approach to compliance optimization was initiated in [K & S, 1986]. The viewpoint of that paper is as follows. The compliance minimization problem can be formulated (before the introduction of composites) as a nonconvex optimization over statically admissible stress fields. The process of relaxation can be viewed in two equivalent ways: as the introduction of optimal composites (the approach we have taken here), or as a construction known as quasiconvexification. The latter requires no explicit mention of composite materials. Rather, one calculates the "rank-one convexification" of the integrand (equivalent to using optimal sequentially laminated composites). Then one tries to prove that the result is "polyconvex" (equivalent to proving a bound on effective moduli via the translation method). If one succeeds, then the result is the correct relaxed integrand.

This program was explored in [K & S, 1986] primarily for scalar problems (electrical resistivity, antiplane shear, etc.) with multiple compliance constraints. However, the problem we consider here – shape optimization for elastic structures under a single load in plane stress – was addressed in Section 5 D of [K & S, 1986] when the basic material A_0 has Poisson's ratio equal to zero (*i.e.* $\kappa = \mu$). In particular, our $F_\lambda(\tau)$ was computed in [K & S, 1986] by the method of quasiconvexification, when the spatial dimension is 2 and $\kappa = \mu$.

With hindsight, we now see that the method of quasiconvexification could have been used to calculate $F_\lambda(\tau)$ for any choice of κ and μ in space dimension two. Indeed, rank-one convexification as in [K & S, 1986] would inevitably lead to the integrand $F_\lambda(\tau)$, since the optimal composites can be rank-two laminates. Moreover, $F_\lambda(\tau)$ is quasiconvex, because it is polyconvex.

One observation from [K & S, 1986] is worth highlighting here. When the boundary condition is hydrostatic, *i.e.* $f = f_0 n$ with f_0 constant, the solution of our optimal design problem is wildly non unique. In fact, there are infinitely many "classical" solutions, obtained by the "concentric sphere construction." The argument is parallel to that of Example 8.3 of [K & S, 1986], so we do not it present here. There is also a "relaxed solution", in which Ω is filled by a homogeneous composite A having uniform density θ, where A and θ achieve

$$\min_{0 \leq \theta \leq 1} \min_{A \in G_\theta} [f_0^2 \langle A^{-1} I, I \rangle + \lambda\theta].$$

The stress field associated to this uniform, relaxed solution is of course $\tau \equiv f_0 I$. Our optimization code, when given such hydrostatic boundary conditions, chooses the homogeneous solution. (In other words, it chooses $\tau \equiv f_0 I$.) The choice is presumably due to a combination of our regularization and the effects of discretization. We wonder whether the code of Bendsoe/Kikuchi/Suzuki might make a different choice.

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