COUPLING THE LEVEL SET METHOD AND THE TOPOLOGICAL GRADIENT IN STRUCTURAL OPTIMIZATION

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Abstract A numerical coupling of two recent methods in shape and topology optimization of structures is proposed. On the one hand, the level set method, based on the classical shape derivative, is known to easily handle boundary propagation with topological changes. However, in practice it does not allow for the nucleation of new holes. On the other hand, the bubble or topological gradient method is precisely designed for introducing new holes in the optimization process. Therefore, the coupling of these two method yields an efficient algorithm which can escape from local minima in a given topological class of shapes. Both have a low CPU cost since they capture a shape on a fixed Eulerian mesh. The main advantage of our coupled algorithm is to make the resulting optimal design more independent of the initial guess.

Keywords: shape and topology optimization, level set method, topological gradient.

1. Introduction

Numerical methods of shape optimization based on the level set method and on shape differentiation make possible topology changes during the optimization process. But they do not solve the inherent problem of ill-posedness of shape optimization which manifests itself in the frequent existence of many local (non global) minima, usually having different topologies. The reason is that the level set method can easily remove holes but can not create new holes in the middle of a shape since the level set function obeys a maximum principle. In practice, this effect can be checked by varying the initialization which yields different optimal shapes with different topologies. This absence of a nucleation mechanism is an inconvenient mostly in 2-d: in 3-d, it is less important since holes can appear by pinching two boundaries.

In [2] we have proposed, as a remedy, to couple our previous method with the topological gradient method (cf. [9][10][11][21][22]). Roughly speaking the topological gradient method amounts to decide whether or not it is favorable to nucleate a small hole in a given shape. As a matter of fact, creating a hole changes the topology and is thus one way of escaping local minima. Our coupled method of topological and shape gradients in the level set framework is therefore much less prone to finding local, non global, optimal shapes. In particular, for most of our 2-d numerical examples of compliance minimization, the expected global minimum is attained from the trivial full domain initialization.

The main contribution of this paper is numeric. We provide a new 2-d numerical example showing that the level set method coupled to the topological gradient can reach an optimum of the objective function, very close to the one obtained by the homogenization method, starting from a trivial initial state. Then a new 3-d example is proposed. Although its solution has a rather complicated topology, it is obtained by the regular level set method, with different initializations, as well as by the coupled method. Thus the introduction of the topological gradient, as already remarked before, is not useful to reach such a complex 3-d solution.

2. Setting of the problem

In this paper we restrict ourselves to linear elasticity. A shape is a bounded open set $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with a boundary made of two disjoint parts $\partial\Omega = \Gamma_N \cup \Gamma_D$, with Dirichlet boundary conditions on Γ_D, and Neumann boundary conditions on Γ_N. All admissible shapes Ω are required to be a subset of a working domain D (a bounded open set of $\mathbb{R}^d$). The shape Ω is occupied by a linear isotropic elastic material with Hooke's law A defined, for any symmetric matrix ξ, by

$$A\xi = 2\mu\xi + \lambda\big(\mathrm{Tr}\xi\big)\,\mathrm{Id},$$

where μ and λ are the Lamé moduli. The displacement field u is the solution of the linearized elasticity system

$$\begin{cases} -\mathrm{div}\,(A\,e(u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \big(A\,e(u)\big)n = g & \text{on } \Gamma_N, \end{cases} \qquad (1)$$

where $f \in L^2(D)^d$ and $g \in H^1(D)^d$ are the volume forces and the surface loads respectively. Assuming that $\Gamma_D \neq \emptyset$, (1) admits a unique solution in $u \in H^1(\Omega)^d$.

The objective function is denoted by $J(\Omega)$. In this paper, only the compliance will be considered:

$$J(\Omega) = \int_\Omega f \cdot u \, dx + \int_{\Gamma_N} g \cdot u \, ds = \int_\Omega A\, e(u) \cdot e(u) \, dx, \qquad (2)$$

where $u = u(\Omega)$ is the solution of (1).

To avoid working on a problem with a volume constraint, we introduce a Lagrange multiplier ℓ and consider the minimization

$$\inf_{\Omega \subset D} \mathcal{L}(\Omega) = J(\Omega) + \ell|\Omega|. \qquad (3)$$

3. Shape derivative

In order to apply a gradient method to the minimization of (3) we recall the classical notion of shape derivative (see e.g. [14], [17], [20], [23]). Starting from a smooth open set Ω, we consider domains of the type $\Omega_\theta = \big(\mathrm{Id} + \theta\big)(\Omega)$, with Id the identity mapping from $\mathbb{R}^d$ into $\mathbb{R}^d$ and θ a vector field in $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. It is well known that, for sufficiently small θ, $(\mathrm{Id} + \theta)$ is a diffeomorphism in $\mathbb{R}^d$.

Definition: The shape derivative of $J(\Omega)$ at Ω is defined as the Fréchet derivative in $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ at 0 of the application $\theta \to J\big((\mathrm{Id} + \theta)(\Omega)\big)$, i.e.

$$J\big((\mathrm{Id} + \theta)(\Omega)\big) = J(\Omega) + J'(\Omega)(\theta) + o(\theta) \quad \text{with} \quad \lim_{\theta \to 0} \frac{|o(\theta)|}{\|\theta\|} = 0 \,,$$

where $J'(\Omega)$ is a continuous linear form on $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

We recall the following classical result (see [4] and references therein) about the shape derivatives for the compliance J.

Theorem 1: *Let Ω be a smooth bounded open set and $\theta \in W^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$. If $f \in H^1(\Omega)^d$, $g \in H^2(\Omega)^d$, $u \in H^2(\Omega)^d$, then the shape derivative of (2) is*

$$\begin{aligned} J'(\Omega)(\theta) = & \int_{\Gamma_N} \left(2 \left[\frac{\partial (g \cdot u)}{\partial n} + H g \cdot u + f \cdot u \right] - A e(u) \cdot e(u) \right) \theta \cdot n \, ds \\ & + \int_{\Gamma_D} A e(u) \cdot e(u) \, \theta \cdot n \, ds, \end{aligned} \qquad (4)$$

where H is the mean curvature defined by $H = \mathrm{div} n$.

4. Topological derivative

One drawback of the method of shape derivative is that there is no change of topology in the parameterization Ω_θ. Numerical methods based on the shape derivative may therefore fall into a local minimum. A remedy to this inconvenience has been proposed as the bubble method, or topological asymptotic method, [10], [11], [22]. The main idea is to test the optimality of a domain to topology variations by removing a small hole with appropriate boundary conditions.

We give a brief review of this method that we shall call in the sequel topological gradient method. Consider an open set $\Omega \subset \mathbb{R}^d$ and a point $x_0 \in \Omega$. Introduce a fixed model hole $\omega \subset \mathbb{R}^d$, a smooth open bounded subset containing the origin. For $\rho > 0$ we define the translated and rescaled hole $\omega_\rho = x_0 + \rho\omega$ and the perforated domain $\Omega_\rho = \Omega \setminus \bar{\omega}_\rho$. The goal is to study the variations of the objective function $J(\Omega_\rho)$ as ρ goes to 0. In the framework of structural optimization we put Neumann boundary conditions on $\partial\omega_\rho$.

Definition: If the objective function admits the following so-called topological asymptotic expansion for small $\rho > 0$

$$J(\Omega_\rho) = J(\Omega) + \rho^d D_T J(x_0) + o(\rho^d),$$

then $D_T J(x_0)$ is called the topological derivative at point x_0.

If the model hole ω is the unit ball, the following result gives the expression of the topological derivative for the compliance $J(\Omega)$ (see [11], [22]).

Theorem 2: *Take ω to be the unit ball of $\mathbb{R}^d$. Assume for simplicity that $f = 0$ and that $g \in H^2(\Omega)^d$ and $u \in H^2(\Omega)^d$. For any $x \in \Omega$ the topological derivative of J is, for $d = 2$,*

$$D_T J(x) = \frac{\pi(\lambda + 2\mu)}{2\mu(\lambda + \mu)} \big\{ 4\mu Ae(u) \cdot e(u) + (\lambda - \mu) tr(Ae(u)) tr(e(u)) \big\}(x), \tag{5}$$

and for $d = 3$,

$$D_T J(x) = \frac{\pi(\lambda + 2\mu)}{\mu(9\lambda + 14\mu)} \big\{ 20\mu Ae(u) \cdot e(u) + (3\lambda - 2\mu) tr(Ae(u)) tr(e(u)) \big\}(x). \tag{6}$$

A straightforward calculation shows that the expressions (5) and (6) are nonnegative. This means that, for compliance minimization, there is no interest in nucleating holes if there is no volume constraint. However, if a volume constraint is imposed, the topological derivative may have negative values due to the addition of the term $-\ell|\omega|$. For the minimization problem (3), the corresponding topological gradient is

$$D_T\mathcal{L}(x) = D_T J(x) - \ell|\omega|.$$

At the points x where $D_T\mathcal{L}(x)$ is negative, we introduce holes into the current domain Ω.

5. Level set method for shape optimization

Consider $D \subset \mathbb{R}^d$ a bounded domain in which all admissible shapes Ω are included, i.e. $\Omega \subset D$. We parameterize the boundary of Ω by means of a level set function, following the idea of Osher and Sethian [16]. We define this level set function ψ in D such that

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap D, \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega, \\ \psi(x) > 0 & \Leftrightarrow x \in \left(D \setminus \overline{\Omega}\right). \end{cases} \tag{7}$$

The normal n to the shape Ω is recovered as $\nabla\psi/|\nabla\psi|$ and the mean curvature H is given by $\operatorname{div}(\nabla\psi/|\nabla\psi|)$.

During the optimization process, the shape $\Omega(t)$ is going to evolve according to a fictitious time parameter $t \in \mathbb{R}^+$ which corresponds to descent stepping. The evolution of the level set function is governed by the following Hamilton-Jacobi transport equation [16]

$$\frac{\partial\psi}{\partial t} + V|\nabla\psi| = 0 \quad \text{in} \quad D, \tag{8}$$

where $V(t,x)$ is the normal velocity of the shape's boundary.

The choice of the normal velocity V is based on the shape derivative computed in Theorem 1

$$\mathcal{L}'(\Omega)(\theta) = \int_{\partial\Omega} v\,\theta\cdot n\,ds, \tag{9}$$

where the integrand $v(u,n,H)$ depends on the state u, the normal n and the mean curvature H. The simplest choice is to take the steepest descent $\theta = -vn$. This yields a normal velocity for the shape's boundary $V = -v$ (remark that v is given everywhere in D and not only on the boundary $\partial\Omega$). Another choice consists in smoothing the velocity field vn by applying the Neumann-to-Dirichlet map to $-vn$ (see e.g. [4], [7], [13]). The method described in details in [12] is used in all the numerical computations of Sections 7 and 8.

The main point is that the Lagrangian evolution of the boundary $\partial\Omega$ is replaced by the Eulerian solution of a transport equation in the whole fixed domain D. Likewise the elasticity equations for the state u (and for the adjoint state p) are extended to the whole domain D by using the so-called "ersatz material" approach.

The Hamilton-Jacobi equation (8) is solved by an explicit second order upwind scheme (see e.g. [18]) on a Cartesian grid. The boundary conditions for ψ are of Neumann type. Since this scheme is explicit in time, its time stepping must satisfy a CFL condition. In order to regularize the level set function (which may become too flat or too steep), we reinitialize it periodically by solving another Hamilton-Jacobi equation which admits as a stationary solution the signed distance to the initial interface [18].

6. Optimization algorithm

For the minimization problem (3) we propose an iterative coupling of the level set method and of the topological gradient method. Both methods are gradient-type algorithms, so our coupled method can be cast into the framework of alternate directions descent algorithms.

The level set method relies on the shape derivative $\mathcal{L}'(\Omega)(\theta)$ of Section 3, while the topological gradient method is based on the topological derivative $D_T\mathcal{L}(x)$ of Section 4. These two types of derivative define independent descent directions that we simply alternate as follows.

In a first step, the level set function ψ is advected according to the velocity $-v$ where v is the integrand in the shape derivative $\mathcal{L}'(\Omega)$, see (9). In a second step, holes are introduced into the current domain Ω where the topological derivative $D_T\mathcal{L}(x)$ is minimum and negative.

In practice, it is better to perform more level set steps than topological gradient steps. Therefore, the main parameter of our coupled algorithm is an integer n_{opt} which is the number of gradient steps between two successive application of the topological gradient Our proposed algorithm is an iterative method, structured as follows:

1 Initialization of the level set function ψ_0 corresponding to an initial guess Ω_0 (usually the full working domain D).

2 Iteration until convergence, for $k \geq 0$:

   (a) **Elasticity analysis.** Computation of the state u_k through one problem of linear elasticity posed in Ω_k. This yields the values of the shape derivative and of the topological gradient.

   (b) **Shape gradient.** If $\mod(k, n_{top}) < n_{top}$, the current shape Ω_k, characterized by the level set function ψ_k, is deformed into a new shape Ω_{k+1}, characterized by ψ_{k+1} which is the solution of the transport Hamilton-Jacobi equation (8) after a time interval Δt_k with the initial condition ψ_k and a velocity $-v_k$ computed in terms of u_k. The time of integration Δt_k is chosen such that $\mathcal{L}(\Omega_{k+1}) \leq \mathcal{L}(\Omega_k)$.

(c) **Topological gradient.** If $\mod(k, n_{top}) = 0$, we perform a nucleation step. We obtain a new shape Ω_{k+1} by inserting new holes into the current shape Ω_k.

For details about the shape gradient step and the topological gradient step, we refer to our previous works [2][4].

7. A numerical example in 2-d

The 2-d example is a variation of the classical cantilever, but its optimal solution seems to have a more complex topology. It consists in a rectangular domain of dimensions 10×8 with a square hole (cf.Figure 1). The hole's boundary is submitted to an homogeneous Dirichlet boundary condition. The domain is meshed with a regular 150×120 grid (16360 elements). Figure 1 shows the composite and penalized solutions obtained by the homogenization method (see [1][5][6]). Since the composite solution is a global optimum of the problem, it will be used as a reference solution. Figure 2 shows the solution obtained by the algorithm coupling shape and topological sensitivity, starting from the full domain, with 1 step of topological gradient every 10 iterations. Figure 4 shows different solutions obtained by the level set algorithm (without topological gradient) for different initial guesses with various number of holes, ranging from 0 to 160.

The convergence history of Figure 6 gives some interesting hints on the efficiency of the level set method without topological gradient: first, it confirms that, of course, a "topologically poor" initialization cannot allow a convergence to a good solution; second, it shows that initializing with "many holes" is not a good idea too. The good initial state is in between, but it is generally not easy to find. The topological gradient allows the convergence to the best solution, starting from the full domain, without the need of adjusting any tricky numerical parameters. Remark that the solution computed from initialization 3 (22 holes) is also good, but it has been reached after an history where it had to escape from many local minima, using the tolerance of the algorithm to (small) increases of the objective function. This comportment is controlled by a numerical parameter that is not easy to tune. Remark also that the best level set solution has a better performance than the penalized solution computed by the homogenized method.

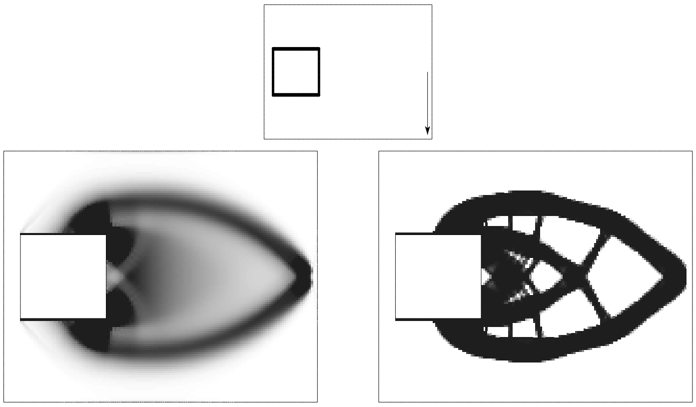

Figure 1. Definition of the 2d problem (above). Homogenization method (bellow): composite (left) and penalized (right) solutions.

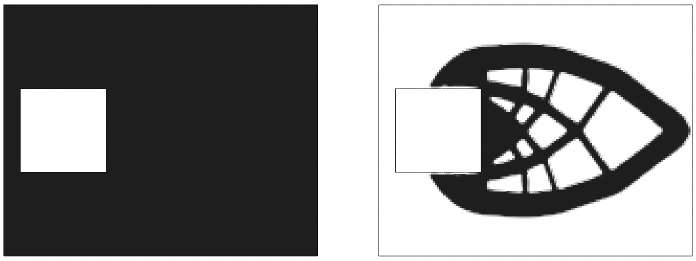

Figure 2. The initial configuration (full domain) and the solution obtained by the level set method with topological gradient.

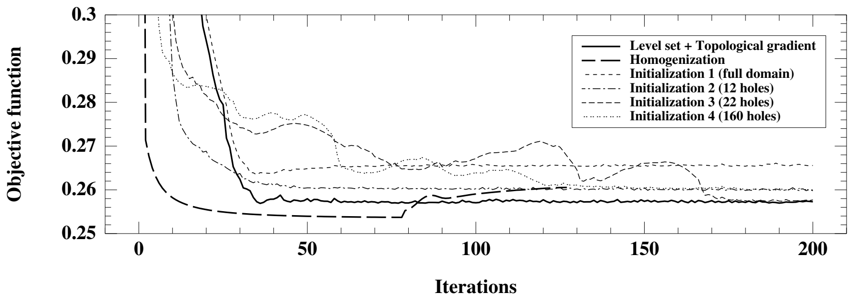

Figure 3. Convergence history of the homogenization method, the level set method with topological gradient (full domain initialization), and the plain level set method with 4 different initial states.

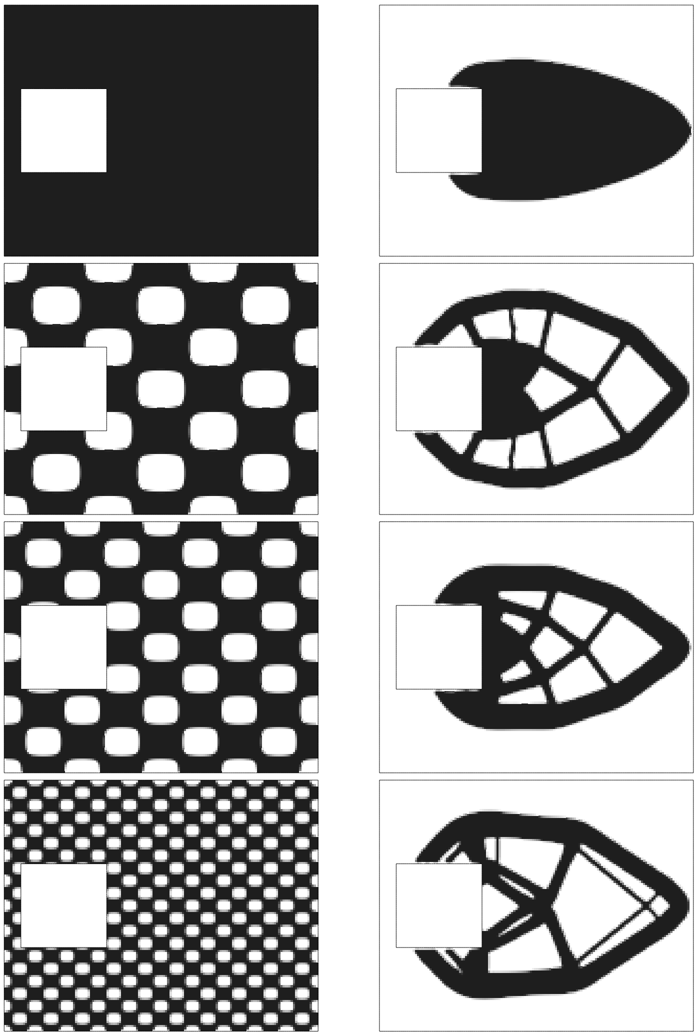

Figure 4. Four solutions obtained by the plain level set method (right) with four different initializations (left): full domain, 12 holes, 22 holes and 160 holes.

8. A numerical example in 3-d

We propose and test-case that have a very topologically complex solution. It is defined by Figure 5 (above). The bottom face is submitted to a uniform Dirichlet boundary condition.

The domain is meshed with 10976 hexaedral elements. The coupled method, level set plus topological gradient every 5 iterations, has been compared to the nominal level set method starting from two initial states (full domain and 8 holes uniformly distributed). The 3 solutions obtained cannot be distinguished on a picture. Figure 5 shows 3 views of the solution and Figure 6 confirms that the objective functions of the converged solutions are very close.

As suspected in [2], the topological gradient seems not to be as efficient and useful in 3-d as it is in 2-d.

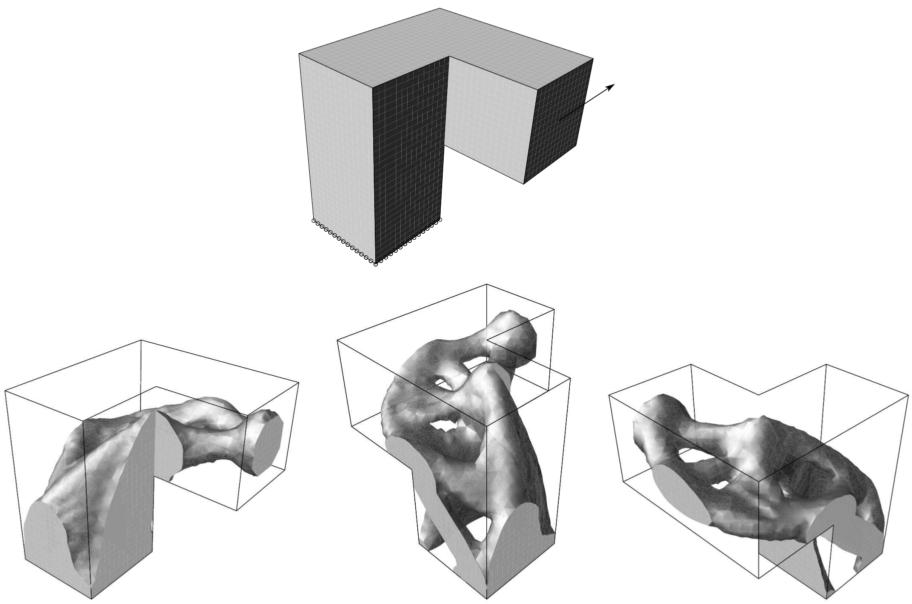

Figure 5. Three different views of the optimal shape obtained for the problem defined above.

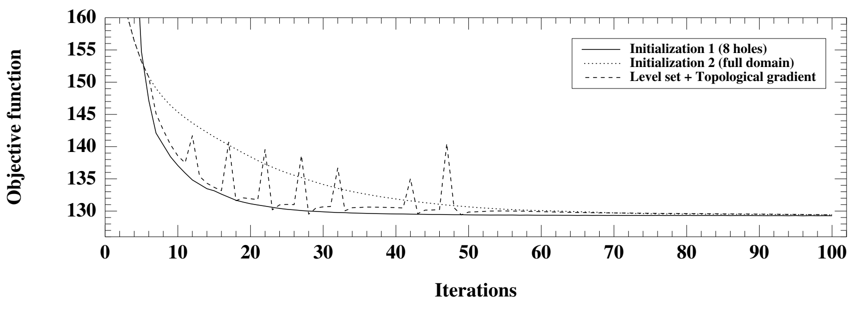

Figure 6. Convergence history of the 3d problem for the plain level set method with two different initializations, and the level set method with topological gradient.

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