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On some recent advances in shape optimization

Grégoire ALLAIRE [a], Antoine HENROT [b]

[a] **Centre de mathématiques appliquées, École polytechnique, 91128 Palaiseau, France**

[b] **École des mines et institut Élie Cartan, B.P. 239, 54506 Vandoeuvre les Nancy cedex, France**
E-mail: allaire@cmapx.polytechnique.fr; henrot@iecn.u-nancy.fr

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Abstract. In this Note we give a short review on recent developements in shape optimization. We explain how the generic non-existence of solutions can be circumvent. Either one can impose some geometric restrictions on the class of admissible domains to get existence (we then explain how to write the usual optimality conditions), or generalized designs are allowed which leads to relaxation by homogenization techniques (we thus obtain topology optimization methods). © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

solids and structures / shape optimization / existence of optimal shapes / domain derivative / relaxation / topology optimization

Sur quelques récents développements de l'optimisation de forme

Résumé. *Dans cette Note, nous donnons un aperçu de quelques développements récents en optimisation de forme. Nous expliquons comment remédier à la non-existence générique de solutions. On peut imposer des restrictions d'ordre géométrique à la classe des domaines admissibles pour obtenir l'existence (nous expliquons alors comment écrire les conditions d'optimalité). On peut aussi permettre des formes généralisées qui conduisent à la relaxation via les techniques d'homogénéisation (méthodes d'optimisation topologique). © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS*

solides et des structures / optimisation de forme / existence de forme optimale / dérivée par rapport au domaine / relaxation / optimisation topologique

1. Position of the problem

1.1. Introduction

Shape optimization amounts to find the optimal shape of a domain which minimizes or maximizes a given criterion (often called an objective function). Here, we are interested in the case where this criterion is computed through the solution of a partial differential equation (the so-called state equation) which makes the optimization problem highly non-trivial. For example, a possible criterion could be the maximal conductivity or rigidity of the domain under some specified loading conditions (and possibly with a volume or weight constraint) with a second-order elliptic partial differential equation as state equation modeling the conductivity or the elasticity of the structure. As such, shape optimization (or optimal design) may be seen as a special branch of distributed control theory, to which most of the previous vocabulary is borrowed (see, e.g., [1]). It has a long history and has been studied by many different methods. There is therefore a vast literature in this field, and we refer the reader to [2–7] and references therein. Our attempt here is to

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make a brief review of some fundamental ideas and recent developments in the field. In such a short note, we can not possibly be exhaustive and the selection of reported topics is, of course, very personal.

The most important difficulty in shape optimization is the generic non-existence of classical solutions (by this we mean well-defined shapes, i.e., domains in the physical space). This is seemingly just a theoretical problem for mathematicians, but it has a dramatic consequence for practical and numerical applications. Indeed, most algorithms are not convergent under mesh refinement or highly sensitive to initial guesses (these are the usual consequences of non-existence) which implies that the result of a computation is never guaranteed to be optimal, even approximately. There are therefore two possible routes to avoid this serious inconvenient. The first one is to restrict the class of admissible designs by adding further constraints which ensures the existence of an optimum. This point of view is developed in Section 2, the main question being *what kind of restrictions must be put on the class of admissible domains in order to have existence*? Section 3 then deals with the corresponding optimality conditions. On the contrary, the second one is to enlarge the class of admissible designs by allowing for generalized designs for which there are optimal solutions. This alternative point of view (called relaxation) is discussed in Section 4. The main goal of this note is therefore to show how one can go beyond the pessimistic statement: *a shape optimization problem has generally no classical solution* which is merely valid at first sight.

1.2. Non-existence of optimal shapes

Counter-examples to the existence of optimal designs were devised by Murat [8] (numerical or analytical evidence were also found in [9,10]). Let us give a simple example of a situation where no (regular) optimal domain exists (it has been suggested by G. Buttazzo, but a similar one proposed by Murat is described in [5]). It has also the advantage to give a practical way to prove non existence in typical situations.

Let D be a fixed domain, for every open set $\omega \subset D$, we solve the Dirichlet problem:

$$\begin{cases} -\Delta u_\omega = f & \text{in } \omega \\ u_\omega = 0 & \text{on } D \setminus \omega \end{cases}$$

We want to minimize the functional:

$$J(\omega) = \int_D (u_\omega - u_0)^2 \, \mathrm{d}x$$

where u_0 is a given function in $L^2(D)$. A physical interpretation of the problem could be the following: D is a box heated by a heat source f and $D \setminus \omega$ is the place filled up with ice. We want to find the best location for the freezing zone in order to approach in a best way an ideal temperature given by u_0.

Let us consider a very simple configuration to show that, in general, this problem has no solution. We choose $f = 1$, $u_0 = c = \text{constant}$ and $D =$ is the unit ball in $\mathbb{R}^2$.

According to maximum principle, for every $\omega \subset D$, $0 \leqslant u_\omega \leqslant u_D = (1 - r^2)/4 \leqslant 1/4$ therefore if $c \geqslant 1/4$ we have $u_\omega - c \leqslant u_D - c \leqslant 0$ and then:

$$J(\omega) = \int_D (u_\omega - c)^2 \, \mathrm{d}x \geqslant \int_D (u_D - c)^2 \, \mathrm{d}x = J(D)$$

what proves that $\omega = D$ realizes the minimum of J (it is logical: since the desired temperature is high, we have to freeze the less possible in this case).

Let us now consider the case $0 < c < 1/8$. It is easy to see, by a straightforward calculation, that we can always find a ball of center 0 and radius $R < 1$ which gives a better value for J than D.

Let us now prove, by contradiction, that J cannot have a (regular) minimizer. If Ω would be such a minimizer, let B_ε a ball of (small) radius ε included in $D \setminus \overline{\Omega}$. Let us introduce $\Omega_\varepsilon = \Omega \cup B_\varepsilon$ and let us prove that, for ε small enough, Ω_ε is a 'better' domain than Ω. Since Ω_ε has two disjoint connected components, the solution u_{Ω_ε} can be computed separately on each components. Now, on Ω, we have obviously $u_{\Omega_\varepsilon} = u_\Omega$, while on B_ε, u_{Ω_ε} can be computed explicitly (it is radially symmetric). In particular, it is easy to see that, for ε small enough, we have $0 < u_{\Omega_\varepsilon} < c$ on B_ε. Let us now compare $J(\Omega_\varepsilon)$ to $J(\Omega)$. We have:

$$\begin{aligned} J(\Omega_\varepsilon) &= \int_{\Omega_\varepsilon} (u_{\Omega_\varepsilon} - c)^2 \,\mathrm{d}x + \int_{D\setminus\Omega_\varepsilon} c^2 \,\mathrm{d}x = \int_{\Omega} (u_\Omega - c)^2 \,\mathrm{d}x \\ &\quad + \int_{B_\varepsilon} (u_{\Omega_\varepsilon} - c)^2 \,\mathrm{d}x + \int_{D\setminus\Omega} c^2 \,\mathrm{d}x - \int_{B_\varepsilon} c^2 \,\mathrm{d}x = J(\Omega) + \int_{B_\varepsilon} \big((u_{\Omega_\varepsilon} - c)^2 - c^2\big)\mathrm{d}x \end{aligned}$$

Now, as soon as $0 < u_{\Omega_\varepsilon} < c$, we have $(u_{\Omega_\varepsilon} - c)^2 < c^2$ and therefore $J(\Omega_\varepsilon) < J(\Omega)$. That proves the result.

Even if there is no minimum, it is interesting to observe (at least numerically) the behaviour of minimizing sequences for the previous example. What can be seen is that the freezing area $D \setminus \omega$ wants to have more and more small components in order to reach **by an homogenization process** a temperature the closer to the small constant c. We will come back on this situation in Section 4. We also immediately understand that, if we want to have existence in such an example, we have to prevent this behaviour by imposing some geometric restrictions on the class of admissible domains (see Subsections 2.1–2.3).

2. Existence results of classical solutions

2.1. Uniform regularity constraint of the boundary

A first natural idea to prevent oscillations and/or scattering of the minimizing sequences consists in working with domains which have some uniform regularity. This regularity can be expressed either in term of uniform Lipschitz regularity of the boundary or in term of uniform cone condition. Let us choose this description which is more geometric.

DEFINITION 2.1. – Let y be a point in $\mathbb{R}^N$, ξ a unit vector and $\varepsilon > 0$. We will denote by $C(y, \xi, \varepsilon)$ the cone defined by:

$$C(y,\xi,\varepsilon) = \big\{z \in \mathbb{R}^N,\ (z-y,\xi) \geqslant \cos(\varepsilon)|z-y| \text{ and } 0 < |z-y| < \varepsilon\big\}$$

Then, we will say that an open set Ω has the ε-cone property if:

$$\forall x \in \partial\Omega,\ \exists \xi_x \text{ unit vector such that } \forall y \in \overline{\Omega} \cap B(x,\varepsilon) \quad C(y,\xi_x,\varepsilon) \subset \Omega$$

It can be proved, see, e.g., [11] or [4], that the previous property is equivalent to uniform Lipschitz property of the boundary of Ω.

In the sequel (and also in Sections 2.2 and 2.4), we will consider open sets contained in a fixed ball B. We now introduce the following class of admissible domains:

$$\mathcal{O}_\varepsilon = \{\Omega \text{ open set},\ \Omega \subset B,\ \Omega \text{ has the } \varepsilon\text{-cone property}\}$$

Then, we are able to prove existence of a solution for a large class of shape optimization problems with admissible domains varying in $\mathcal{O}_\varepsilon$. That was done in 1975–76 by D. Chenais, see [11]. Let us now give a precise statement.

Let $f \in L^2(B)$ and for all $\omega \subset B$, let us introduce the solutions u_ω and v_ω of the following Dirichlet and Neumann problems:

$$\begin{cases} -\Delta u_\omega = f & \text{in } \omega \\ u_\omega = 0 & \text{on } \partial\omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v_\omega + v_\omega = f & \text{in } \omega \\ \dfrac{\partial v_\omega}{\partial n} = 0 & \text{on } \partial\omega \end{cases} \tag{1}$$

We also consider a function $j : B \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ continuous and such that there exists a constant C with

$$\forall x \in B,\ r \in \mathbb{R},\ p \in \mathbb{R}^N \quad \left|j(x,r,p)\right| \leqslant C\left(1 + r^2 + |p|^2\right) \tag{2}$$

At last, we assume that the functionals we want to minimize are given by (for $\omega \subset B$):

$$J_1(\omega) = \int_\omega j\left(x, u_\omega(x), \nabla u_\omega(x)\right) \mathrm{d}x \tag{3}$$

or

$$J_2(\omega) = \int_\omega j\left(x, v_\omega(x), 0\right) \mathrm{d}x + \alpha \int_\omega \left|\nabla v_\omega(x)\right|^2 \mathrm{d}x \tag{4}$$

with $\alpha \geqslant 0$. Let us remark that the above functional are well defined according to (2). As an example, we can consider least square functionals, as the introducing counter-example, if we make the particular choices $j(x,r,p) = (r - g(x))^2$ (with g a given continuous function on $\overline{B}$) or $j(x,r,p) = |p - p_0(x)|^2$ with $p_0 : B \to \mathbb{R}^N$ is continuous and bounded. Let us also remark that the formulation allows cases where the integration is done only in a subdomain of ω: it suffices to introduce the characteristic function of this subdomain in j.

We want to minimize functionals defined by (3) or (4) either on the whole class $\mathcal{O}_\varepsilon$ or on a subclass of sets with given measure. We can state:

THEOREM 2.2 (Chenais). – *Let $\mathcal{O}_{ad}$ denote the class of admissible domains given by $\mathcal{O}_{ad} = \mathcal{O}_\varepsilon$ (for a fixed ε) or by $\mathcal{O}_{ad} = \{\omega \in \mathcal{O}_\varepsilon,\ |\omega| = d\}$. Let j be a function satisfying* (2). *We assume moreover that the functional J_1 (resp. J_2) defined by* (3) (*resp.* (4)) *is estimated from below. Then, there exists at least one domain Ω in $\mathcal{O}_{ad}$ which minimizes J_1 (resp. J_2).*

The proof relies on the use of the Hausdorff convergence for open sets, together with a result of continuity with respect to domain variations in the class $\mathcal{O}_\varepsilon$, see the above mentioned references.

2.2. Limited number of connected components

In two dimensions, we can state a remarkable result, due to V. Šverak, see [12], which gives existence with a much weaker constraint on the domains. This constraint is of topological nature since we assume now that **the number of connected components of the complementary** must stay bounded. Let us be more precise. Let l be an integer $\geqslant 1$, for every open set $\omega \subset B$ we will denote by $\sharp\omega^c$ the number of connected components of the complementary of ω. We then define the set:

$$\mathcal{O}_l = \left\{\omega \subset B,\ \sharp\omega^c \leqslant l\right\}$$

We can prove:

THEOREM 2.3 (Šverak). – *Let $\mathcal{O}_l$ denotes the class of admissible plane domains defined just above. Let j be a function satisfying* (2) *and J_1 a functional defined by* (3). *We assume J_1 to be estimated from below. Then, there exists at least one domain Ω in $\mathcal{O}_l$ which minimizes J_1.*

2.3. Perimeter constraints

There are several cases where imposing constraints on the perimeter of the admissible domains appear very natural. This is the case, in particular, when the perimeter appears directly in the functional we want to minimize as a surface tension term. It can also be a good measure of the cost of the design we want to construct. We will see that the boundedness of the perimeter can give some compactness property which is very useful to get existence results (see, e.g., [13]). For that purpose, we need to work with the generalized perimeter introduced by De Giorgi, see, e.g., [14]. We recall that it can be defined as $P(\Omega) = \|\nabla\chi_\Omega\|_1$ that is $P(\Omega)$ is the total variation of the gradient of the characteristic function considered as a Radon measure. We will denote by $\mathcal{M}_b(D)$ the set of Radon measures in an open set D. Now, the compactness embedding of the set $BV(D) = \{f \in L^1(D);\ \nabla f \in \mathcal{M}_b(D)^N\}$ into $L^1(D)$ allows us to prove the following result:

THEOREM 2.4. – *Let Ω_n be a sequence of measurable sets in an open set D in $\mathbb{R}^N$. We assume that*:

$$|\Omega_n| + P(\Omega_n) \leqslant C \quad \textit{independent of } n$$

Then, there exists $\Omega \subset D$ measurable and a subsequence Ω_{n_k} such that:

$$\chi_{\Omega_{n_k}} \longrightarrow \chi_\Omega \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}^N) \quad \textit{and} \quad \nabla\chi_{\Omega_{n_k}} \overset{*}{\rightharpoonup} \nabla\chi_\Omega \quad \textit{in } \sigma\big(\mathcal{M}_b(D), C_0(D)\big)$$

Moreover, if D has finite measure, the convergence of $\chi_{\Omega_{n_k}}$ to χ_Ω takes place in $L^1(D)$.

As a consequence, we can prove existence results in some situations like the followings.

We assume that we have a state equation (a p.d.e. for example) which associates to every measurable set Ω a function $y(\Omega) \in L^2(D)$ and that this construction satisfies:

$$\chi_{\Omega_n} \xrightarrow{L^1} \chi_\Omega \quad \Longrightarrow \quad y(\Omega_n) \longrightarrow y(\Omega) \text{ in } L^2(D) \tag{5}$$

For $f \in L^2(D)$ and $\varepsilon > 0$ being given, we define the functional

$$J(\Omega) = \int_D \big(y(\Omega) - f\big)^2\,\mathrm{d}x + \varepsilon P(\Omega) \tag{6}$$

(ε is for example a constant of superficial tension). Let us now consider the two problems:

$$\min\big\{J(\Omega),\ \Omega \subset D\big\} \quad \text{where } D \text{ has a finite measure} \tag{7}$$

$$\min\big\{J(\Omega),\ \Omega \subset D,\ |\Omega| = V_0\big\} \quad \text{with } V_0 \text{ fixed} \tag{8}$$

Then, we have:

THEOREM 2.5. – *Problems* (7) *and* (8) *have a solution.*

2.4. Monotone functional

In this section, we are going to give an existence result when the functional we want to minimize is non increasing with respect to set inclusion:

$$J(A_1) \geqslant J(A_2) \quad \text{if } A_1 \subset A_2 \subset B \tag{9}$$

where B is a ball which is fixed in the sequel. This result is due to G. Buttazzo and G. Dal Maso, see [15]. It is particularly interesting for problems involving eigenvalues, which often have this monotonicity property. In order not to enter into technicalities, we give a statement of their theorem which is not completely precise, referring to [15] or [4] for more details. Nevertheless, we need to introduce the notion of γ-convergence.

DEFINITION 2.6. – Let ω_n a sequence of domains. We say that ω_n γ-converges to a domain ω if, for every function $f \in L^2(B)$, the solutions u_{ω_n} of the Dirichlet problem (1) set on ω_n converge in the Sobolev space $H^1_0(B)$ to u_ω solution of (1) for ω.

In the previous definition, all functions can be considered as elements of $H_0^1(B)$ thanks to an extension by zero outside ω_n or ω.

It can be proved using the maximum principle, see [12] or [4], that it is sufficient to consider the right-hand side $f = 1$ in the previous definition.

We can now state the existence result:

THEOREM 2.7 (Buttazzo–Dal Maso). – *Let J be a functional defined on domains which satisfy*:
(i) *J is lower-semi continuous for γ-convergence*;
(ii) *J is non-increasing with respect to set inclusion*;
Then, for every real number c between 0 and $|B|$, the problem:

$$\min\{J(\omega);\ \omega \subset B,\ |\omega| = c\}$$

has a solution.

As a consequence, interesting results can be proved for functionals involving eigenvalues of the Laplacian with Dirichlet boundary condition. For example, if $\Phi : \mathbb{R}^m \to \mathbb{R}$ is a non-decreasing, lower semi-continuous function, then the problem

$$\min\{\Phi(\lambda_{k_1}(\omega), \lambda_{k_2}(\omega), \ldots, \lambda_{k_m}(\omega));\ \omega \subset B,\ |\omega| = c\}$$

has a solution. We also refer to [16] and [17] for numerical and theoretical results in this direction.

In the same way, the maximum principle can be used to prove monotonicity of particular functionals and therefore similar existence results are availables for functionals involving the solution of Dirichlet problems.

3. Optimality conditions

3.1. Differentiation with respect to the domain

It seems not necessary to emphasize the interest of optimality conditions in general optimization problems. The difficulty, here, is due to the fact that the set of admissible shapes is not a vector space, so the usual optimality conditions are not available. There are different ways to circumvent this difficulty and the reader can look at standards works like, for example, [18] or [7] to have a good overview on the topic.

In this section, we choose a simple approach. We are only interested in the so-called 'Gateaux-differentiability' of the functionals. This concept is weaker, but it generally gives enough information and good optimality conditions.

Now, let us fix the notations. We want to differentiate a functional J defined for every regular (enough) domains. Let Ω be a such domain in $\mathbb{R}^N$. We will assume, for example, Ω to be of class C^1. Differentiability results are available with less regularity, but it is not the purpose here to discuss this question, we refer to the above-mentioned papers or to [4] for precisions. First of all, we need to describe how we move the domain Ω. Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a C^1 vector field. There are two traditional ways to perturb Ω according to V:

Perturbation of identity. For t small, we know that $\mathrm{Id} + tV$ is a C^1 diffeomorphism, and we set $\Omega_t := (\mathrm{Id} + tV)(\Omega)$.

Speed method. For every point $M \in \mathbb{R}^N$, we consider the differential system:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} X_M(t) = V(X_M(t)) \\ X_M(0) = M \end{cases}$$

Then, Ω_t is defined as the set of points $X_M(t)$ when M covers Ω.

These two methods give the same formulae and we do not discuss here the merit of one or the other. For sake of simplicity, we choose in the sequel the method of Perturbation of identity. Then, it is natural to define the (Eulerian or Gateaux) derivative of the functional J at the point Ω in the direction V to be:

$$\mathrm{d}J(\Omega, V) := \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t} \tag{10}$$

Since the functional we have to consider generally depends on the solution of some p.d.e. (the state function), we also have to define the differentiability of such solutions with respect to the parameter t. Let u denotes the state function on Ω and u_t the state function on Ω_t. We also have two ways to define the derivative of u_t:

Material derivative. We transport the situation on the fixed domain Ω by the change of variable defined by the transformation $\mathrm{Id} + tV$. Then, we look at the differentiability of the map $t \to u_t \circ (\mathrm{Id} + tV)$ (for example into the Sobolev space $H^1(\Omega)$). If this map is differentiable for $t = 0$, we will denote by $\dot{u}$ or $\dot{u}(\Omega, V)$ its derivative which is called the material derivative.

Shape derivative. We define here the shape derivative in a local sense. Let $\omega \Subset \Omega$ a fixed open set strictly included in Ω. Then, by definition, we will have $\omega \subset \Omega_t$ for t small enough. Therefore, the state function u_t is well defined on ω and it is convenient to look at the following differential quotient limit which involves the restrictions to ω:

$$\lim_{t \to 0} \frac{u_t - u}{t}$$

If this limit exists for every ω, it defines a function in the whole domain Ω which is denoted by u' or $u'(\Omega, V)$ and which is called the shape derivative of u.

There exists a simple link between these two notions of derivatives. Indeed, it can be proved (by writing $u_t - u = u_t - u_t \circ (\mathrm{Id} + tV) + u_t \circ (\mathrm{Id} + tV) - u$) the following equality:

$$u'(\Omega, V) = \dot{u}(\Omega, V) - V \cdot \nabla u \tag{11}$$

In the sequel, we first give two fundamental formulae which show how to differentiate integrals either on the domain or on its boundary. Then, we explain how to use these formulae to get any other result. In particular, formulae for the derivative of solutions of p.d.e. will be simply obtained from the variational formulation.

3.2. Hadamard integral formula

The fundamental formulae (sometimes called 'Hadamard formulae') which allow to compute most derivatives are the following:

THEOREM 3.1. – *Let Ω be a C^1 domain in $\mathbb{R}^N$ and V a C^1 vector field. Let F a function in the space $C^1((0,\varepsilon), C^0(\overline{\Omega_t})) \cap C^0((0,\varepsilon), C^1(\overline{\Omega_t}))$. Then the functional defined by*:

$$J_1(t) = \int_{\Omega_t} F(t, X)\,\mathrm{d}X \tag{12}$$

is differentiable and its derivative is given by:

$$\begin{aligned} \mathrm{d}J_1(\Omega_t, V) &:= \int_{\Omega_t} \left(\frac{\partial}{\partial t} F(t, X) + \mathrm{div}\big(F(t, X) V(X)\big) \right) \mathrm{d}X \\ &= \int_{\Omega_t} \frac{\partial}{\partial t} F(t, X)\,\mathrm{d}X + \int_{\partial \Omega_t} F(t, \sigma)\, V(\sigma) \cdot n\,\mathrm{d}\sigma \end{aligned} \tag{13}$$

The formula (13) can, of course, be used to compute second derivatives. We now give a formula for functionals defined on the boundary:

THEOREM 3.2. – *Let Ω be a C^2 domain in $\mathbb{R}^N$ and V a C^2 vector field. Let G a function in the space $C^1((0,\varepsilon), C^0(\overline{\Omega_t})) \cap C^0((0,\varepsilon), C^1(\overline{\Omega_t}))$. Then the functional defined by*:

$$J_2(t) = \int_{\partial\Omega_t} G(t,\sigma)\,\mathrm{d}\sigma \tag{14}$$

is differentiable and its derivative is given by:

$$\mathrm{d}J_2(\Omega_t, V) := \int_{\partial\Omega_t} \frac{\partial}{\partial t} G(t,\sigma)\,\mathrm{d}\sigma + \int_{\partial\Omega_t} \left(H(\sigma)G(t,\sigma) + \frac{\partial G(t,\sigma)}{\partial n} \right) V(\sigma)\cdot n\,\mathrm{d}\sigma \tag{15}$$

where in (15) $H(\sigma)$ *is the mean curvature of the boundary at the point* σ *and* $\partial G(t,\sigma)/\partial n$ *the usual normal derivative.*

3.3. Application

Let us now explain how we can use formula (13) in a practical way. Let us assume, for example, that we want to differentiate the functional

$$J(\Omega) := a\int_\Omega |\nabla u_\Omega - \nabla v_0|^2\,\mathrm{d}x + b\int_\Omega |u_\Omega - v_1|^2\,\mathrm{d}x \tag{16}$$

where a and b are real numbers, v_0 (resp. v_1) is a given function in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ (resp. $L^2_{\mathrm{loc}}(\mathbb{R}^N)$) and u_Ω is the solution of the Dirichlet problem (1)

A simple use of formula (13) gives (we assume enough regularity, e.g., Ω of class C^2 and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^N)$):

$$\begin{aligned}\mathrm{d}J(\Omega; V) = 2a\int_\Omega (\nabla u - \nabla v_0)\cdot\nabla u'\,\mathrm{d}x + 2b\int_\Omega (u - v_1)u'\,\mathrm{d}x \\ + a\int_{\partial\Omega} |\nabla u - \nabla v_0|^2\, V\cdot n\,\mathrm{d}x + b\int_{\partial\Omega} |v_1|^2\, V\cdot n\,\mathrm{d}x\end{aligned} \tag{17}$$

Now, we have to precise how we compute u' in this case. We write the variational formulation of the Dirichlet problem on Ω_t:

$$\forall v \in H^1_0(\Omega_t) \qquad \int_{\Omega_t} \nabla u_t\cdot\nabla v\,\mathrm{d}x = \int_{\Omega_t} f v\,\mathrm{d}x \tag{18}$$

Let us fix a function $\varphi \in \mathcal{D}(\Omega)$ (C^∞ with compact support in Ω). Obviously, φ will be in the Sobolev spaces $H^1_0(\Omega_t)$ for t small enough, so we can differentiate equality (18) (with $v = \varphi$ fixed) according to formula (13) to get:

$$\int_\Omega \nabla u'\cdot\nabla\varphi\,\mathrm{d}x + \int_{\partial\Omega} \nabla u\cdot\nabla\varphi V\cdot n\,\mathrm{d}\sigma = \int_{\partial\Omega} f\varphi V\cdot n\,\mathrm{d}\sigma \tag{19}$$

Now, φ being zero in a neighbourhood of the boundary, the two boundary integrals vanish and we infer that u' solves

$$\Delta u' = 0 \quad \text{in } \Omega$$

(at least in the distributional sense). Now, to recover the boundary condition, we remember the equality $u'(\Omega, V) = \dot{u}(\Omega, V) - \nabla u\cdot V$. Since the function $u_t \circ (\mathrm{Id} + tV)$ defined on the fixed domain Ω vanishes on the boundary of Ω for every t (u_t satisfies a homogeneous Dirichlet condition on Ω_t), we deduce, that

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t \circ (\mathrm{Id} + tV)\bigg|_{t=0} = \dot{u}(\Omega, V) = 0 \quad \text{on } \partial\Omega$$

In other words, since $u_t \circ (\mathrm{Id} + tV)$ belongs to the Sobolev space $H_0^1(\Omega)$ for every t, so does the material derivative $\dot{u}(\Omega, V)$. Therefore, according to (11), u' satisfies:

$$u' = -\nabla u \cdot V = -\frac{\partial u}{\partial n} V \cdot n \quad \text{on } \partial\Omega$$

(the last equality coming from the fact that the gradient of u is normal to the boundary).

In the case of more complicated boundary conditions satisfied by u, the boundary conditions satisfied by u' generally come directly from the variational formulation like (19), see [4] and [7].

4. Relaxation and topology optimization

4.1. Basic ideas

The main idea of relaxation (which goes back at least to Young [19]) is to apply the direct method of the calculus of variations, i.e., to establish the convergence of the minimizing sequences. There are two obstacles: the design space may not be compact, and the objective function may not be (lower semi-) continuous. Therefore, relaxation requires the knowledge of the closure of the space of admissible designs and of the extension of the objective function to this closure. This is not an obvious task and a complete answer is known only in a few cases. In the next subsection we show that homogenization is a key tool to obtain this relaxation, i.e., to devise a notion of generalized designs which makes the optimization problem well-posed without changing its physical relevance. As a consequence, numerical computations of relaxed or generalized designs are more stable and efficient.

Another motivation for introducing a relaxation is the difficulty or making variations of a domain as explained in Section 3. In particular, the above variations are too restrictive since they never change the topology of the domain. Relaxation yields new optimality conditions which take into account topology changes, and, in practice, furnishes new numerical algorithms that are very efficient for shape and topology optimization. This explains the interest and success of this approach, both from a theoretical, as well as numerical, standpoint.

4.2. The homogenization approach

The homogenization method in optimal design was initiated by Murat and Tartar [20], Cherkaev and Lurie [21,22], Kohn and Strang [23]. The numerical efficiency of this method in realistic problems of shape optimization was later demonstrated by Bendsoe and Kikuchi [24–26].

Let us explain briefly this method on a model example (see [2] for more details). The goal is to find a shape of maximal rigidity and minimal weight. We consider a bounded reference domain $\Omega \in \mathbb{R}^N$ ($N = 2$ or 3), subject to given boundary conditions (for example, a surface loading f on part of its boundary $\partial\Omega_L$ and zero displacement on another part $\partial\Omega_D$). An admissible design ω is a subset of the reference domain Ω obtained by removing one or more holes (the new boundaries created this way are traction-free): it is occupied by a linearly elastic material with isotropic elasticity tensor A (with bulk and shear moduli κ and μ). The equations of linearized elasticity for the resulting structure are:

$$\begin{cases} \sigma = Ae(u), \quad e(u) = (\nabla u + \nabla^t u)/2, \quad \operatorname{div} \sigma = 0 \quad \text{in } \omega \\ u = 0 \quad \text{on } \partial\Omega_D, \quad \sigma \cdot n = f \quad \text{on } \partial\Omega_L, \quad \sigma \cdot n = 0 \quad \text{on } \partial\omega \setminus \partial\Omega \end{cases} \tag{20}$$

A convenient measure of the rigidity of the design ω is its compliance defined by:

$$c(\omega) = \int_{\partial\Omega_L} f \cdot u = \int_\omega Ae(u) \cdot e(u) = \int_\omega A^{-1}\sigma \cdot \sigma \tag{21}$$

Introducing a positive Lagrange multiplier ℓ, the goal is to minimize, over all subsets $\omega \subset \Omega$, the weighted sum $J(\omega)$ of the compliance and the weight (proportional to the volume $|\omega|$), namely to compute

$$\inf_{\omega\subset\Omega} \big(J(\omega) = c(\omega) + \ell|\omega|\big) \tag{22}$$

The Lagrange multiplier ℓ has the effect of balancing the two contradictory objectives of rigidity and lightness of the optimal structure (increasing its value decreases the weight). As already said, in absence of any supplementary constraints on the admissible designs ω, the objective function $J(\omega)$ has no minimizer, i.e., there is no optimal shape. The physical reason is that any admissible design can be improved by replacing a few big holes by many smaller ones. Thus, a minimizing sequence of $J(\omega)$ do not converge to a classical shape, but rather is described, through a limiting procedure of homogenization, by a composite material obtained by micro-perforation of the original material A. Relaxing the problem precisely allows for such perforated composites as generalized admissible designs.

A composite material is parametrized by two functions: $\theta(x)$, its local volume fraction of material taking values between 0 and 1, and $A^*(x)$, its effective elasticity tensor corresponding to its microstructure. By using homogenization theory, one can extend the objective function to these generalized designs: the relaxed objective function is

$$\tilde{J}\big(\theta, A^*\big) = \tilde{c}\big(\theta, A^*\big) + \ell\int_\Omega \theta(x) \tag{23}$$

where $\tilde{c}$ is the homogenized compliance defined by

$$\tilde{c}\big(\theta, A^*\big) = \int_{\partial\Omega_L} f \cdot u = \int_\Omega A^*e(u) \cdot e(u) = \int_\Omega {A^*}^{-1}\sigma \cdot \sigma \tag{24}$$

and u is now the solution of the homogenized problem

$$\begin{cases} \sigma = A^*e(u), \quad e(u) = (\nabla u + \nabla^t u)/2, \quad \operatorname{div}\sigma = 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega_D, \quad \sigma\cdot n = f \quad \text{on } \partial\Omega_L \end{cases} \tag{25}$$

Of course, (25) has to be understood in a generalized sense since A^* is not everywhere coercive (in particular, it vanishes in holes where $\theta = 0$). The minimization of $\tilde{J}$ takes place in the set of all density functions $\theta \in L^\infty(\Omega; [0,1])$ and corresponding homogenized tensors A^*. The following theorem gives the existence of generalized optimal designs (but there is no uniqueness) and their link with minimizing sequences of classical designs.

THEOREM 4.1. – *Problem* (23) *is the correct relaxed formulation of the original problem* (22)*, in the sense that,* (i) *there exists at least one generalized optimal design* (θ, A^*)*,* (ii) *each optimal generalized design is the homogenized limit of a minimizing sequence of classical shapes,* (iii) *the minimal values of the original or homogenized energies coincide*

$$\inf_{\omega\subset\Omega} J(\omega) = \min_{\theta, A^*} \tilde{J}\big(\theta, A^*\big)$$

The relaxed formulation (23) is not fully explicit since the set of all composite materials A^* is unknown. However, by using the optimality conditions of the problem, we can restrict the minimization in A^* to an

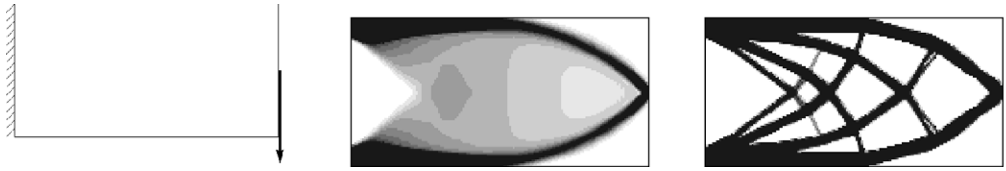

Figure 1. Cantilever problem: boundary conditions (left), relaxed optimal design (middle), penalized optimal design (right).

Figure 1. *Console optimale : conditions aux limites (à gauche), forme optimale relaxée (au milieu), forme optimale pénalisée (à droite).*

explicitly characterized subset, that of so-called sequential laminates. The derivation of these optimality conditions relies on the knowledge of so-called optimal bounds on the effective properties of composite materials. If the stress tensor σ is known, the optimality conditions give explicitly the optimal value of (θ, A^*) in terms of σ (see [25,26] for details).

These results can be generalized to other objective functions J and $\tilde{J}$, namely a sum of several compliances (for multiple loads optimization), or a sum of the first eigenfrequencies. For even more general objective functions, the theory is less complete: the main problem is that we do not know the class of optimal composite materials A^*. Nevertheless, by working only with sequential laminates, it is possible to derive a so-called partial relaxation which is sufficient for numerical purposes (see [2] for details).

4.3. Numerical applications and topology optimization

From a numerical point of view, the main interests of the homogenization method are that it yields shape-capturing algorithms, where the shape is captured on a fixed mesh (while classical numerical algorithms are shape-tracking algorithms where the shape fits to the mesh which is deformed during the optimization process), and that it allows for topology changes through the iterations (which is impossible with more classical algorithms).

For the above example (23), as well as for any objective function involving a sum of compliances or eigenfrequencies, the most efficient numerical algorithms are of the type of optimality criteria (see, e.g., [6]). In practice, the optimization process reduces to an iterative method which computes the solution u of the linear elasticity problem (25) with the previous design parameters, and then update the design parameters (θ, A^*) by using the explicit optimality conditions. This algorithm can be proved to be convergent in some cases, and is always much faster than any gradient type method. On the other hand, for more general objective functions they do not work at all, and a usual gradient method has to be used (which requires the computation of a so-called adjoint state). For all details of implementation and convergence, we refer to [2] and [3].

Of course, any algorithm based on such a relaxed or homogenized formulation produces optimal designs that are truly composite materials and not classical shapes. Since the practical goal is to find a real shape, i.e., a density θ taking only the values 0 or 1, this drawback can be avoided through a post-processing technique that penalizes composite regions. This penalization is performed as follows: a few more iterations of the previous algorithm are run with a modified update of the density which is forced to take values close to 0 or 1.

4.4. Other topological methods

There are other methods in topology optimization which are not based on relaxation. Let us discuss six of them. The first one is based on the idea of convexification or, equivalently, on so-called fictitious materials (see, e.g., [2,3,6] and references therein). This method is often called SIMP (simplified isotropic material with penalization). Convexification retains the notion of material density but forget about microstructure

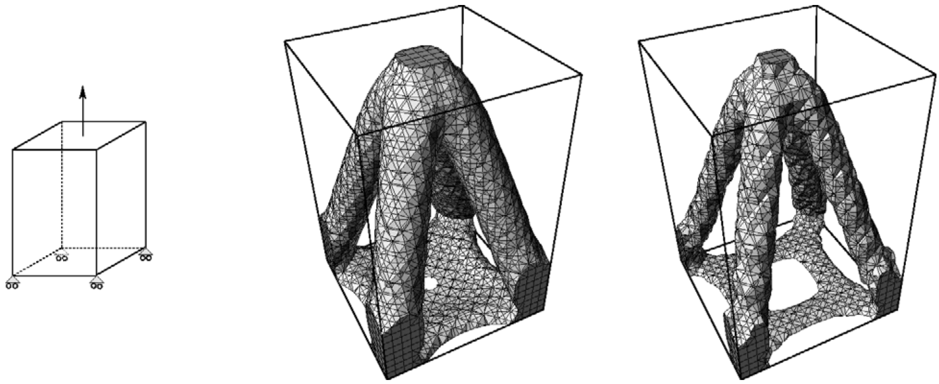

Figure 2. Footstool problem: boundary conditions (left), relaxed optimal design (middle), penalized optimal design (right).

Figure 2. *Tétrapode : conditions aux limites (à gauche), forme optimale relaxée (au milieu), forme optimale pénalisée (à droite).*

and homogenization. The generalized material parameters are simply taken proportional to the density θ. In 2-D, it can be seen as a 'variable thickness' plate. This approach is then coupled with the same penalization procedure as before. The advantages are the following: topology optimization is reduced to sizing optimization, implementation is simple and straightforward, no knowledge of homogenization or composite materials is required. The drawbacks are the lack of connection with the original formulation (a convexified design is not the limit of a sequence of classical designs). As a consequence the convexification method is much more sensitive to the penalization and, although comparable results are produced in some test cases, it can produce worse designs than those of the homogenization (or relaxed) method.

The second method is a generalization of the previous one, called free material design (see, e.g., [3,27]). In this approach, the full elasticity tensor is a design parameter. There is no more any notion of material density or weight, so instead a global resource constraint on the Frobenius norm of the elasticity tensor is imposed.

The third method relies on global stochastic optimization by using, for example, genetic algorithms, simulated annealing, or neural networks (see, e.g., [28]). The idea is to discretize the original problem (22) without any modification. The computing domain is simply discretized in cells which can only be pure material or void. Of course, the problem is ill-posed and any numerical method based on local search is bound to fail miserably. However, global stochastic search of an optimum can avoid the host of local minima. This type of algorithms are very simple to implement, even for complicated problems, but are unfortunately quite expensive because they require many evaluations of the objective function (each of them requiring a finite element computation). They should be used if no other method is available.

The fourth approach attempts to change the topology in the classical framework of shape sensitivity (see, e.g., the bubble method [29], or the topology gradient [30,31]). The main idea is to evaluate the variation of the objective function $J(\omega)$ when a small hole is cut into the admissible shape ω. This type of variation of the domain is not covered by the Hadamard method as described in Section 3, and it yields a new concept of topological derivative. Basically it indicates where a small hole should be put in the shape. This idea allows to obtain new optimality conditions (inside the domain) and is quite useful for numerical purposes.

The fifth method is based on the level set method (see [32,33]). It still works in the classical framework of shape sensitivity, but the boundary (represented as the zero-level set of a function) is captured on an Eulerian fixed mesh. Therefore, it allows for possible topology changes.

Finally, the sixth method, recently introduced in [34,35], is a material distribution optimization where the mass is allowed to concentrate on lower dimensional structures (such as bars, trusses or surfaces).

This formulation has beautiful links with the well-known Monge–Kantorovich problem of optimal mass transportation and the Michell truss problem.

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