OPTIMAL DESIGN WITH SMALL CONTRAST

Grégoire Allaire and Sergio Gutiérrez
[1] *CMAP, Ecole Polytechnique, 91128 Palaiseau, France*
[2] *CMAP, Ecole Polytechnique, 91128 Palaiseau, France*
gregoire.allaire@polytechnique.fr, sergio@cmap.polytechnique.fr

Abstract This paper is concerned with optimal design problems with a special assumption on the coefficients of the state equation. Namely we assume that the variations of these coefficients have a small amplitude. Then, making an asymptotic expansion up to second order with respect to the aspect ratio of the coefficients allows us to greatly simplify the optimal design problem. By using the notion of H-measures we are able to prove general existence theorems for small amplitude optimal design and to provide simple and efficient numerical algorithms for their computation. A key feature of this type of problems is that the optimal microstructures are always simple laminates.

Keywords: Optimal design, homogenization, H-measures.

1. Introduction

Shape or structural optimization is a very active research topic in applied mathematics, which has seen a burst of new ideas in the last twenty years. A common feature of most of the recently developed methods is to try to circumvent the inceptive ill-posedness of shape optimization problems which manifests itself, in numerical practice, by the occurrence of many local minima, possibly far from being global. Probably the most successful approach is the homogenization method [1], [5], [6], [15], [18]: it allows to find a global minimizer in most instances, at the price of introducing composite materials in the optimal shape (a tricky penalization procedure is required for extracting a classical shape out of it). Unfortunately, the rigorous derivation of the homogenized or relaxed formulation of shape optimization is complete only for a few, albeit important, choices of the objective function (mostly self-adjoint problems like compliances or eigenvalues optimization). This difficulty is not just a mathematical problem, but it is also very restrictive from the point of view of numerical applications. Indeed, there are many non-rigorous approaches to treat general objective functions, usually based on some partial relaxations [1], [3], [7], or ad hoc algorithmic ideas like the SIMP method [5]: none of them is as efficient as the original homogenization method applied to compliance minimization, in the sense that its convergence is neither so smooth, nor so global (the resulting optimum may still depend on the initial guess).

Therefore, many authors have tried to extend the homogenization method to more general objective functions, and in particular to cost functions depending on the gradient of the state (or strain or stress). Although this is a very difficult problem, there has been some results in this direction [4], [9], [12], [13], [17]. The objective of the present paper is also to extend the homogenization method to new objective functions. However, our methodology is quite different: in order to make significant progress, we use a strong simplifying assumption, namely that the two component phases involved in the optimal design have close coefficients or material properties. More precisely we consider two-phase optimal design problems in the context of conductivity or linearized elasticity and we make an asymptotic expansion of the coefficients in terms of the small amplitude parameter that characterizes the variations between the two phases. Restricting ourselves to terms up to second order greatly simplifies the situation. However, the small amplitude optimal design problem is still ill-posed and requires relaxation. The nice feature of our approach is that this relaxation is quite simple because the necessary and delicate tools of homogenization are replaced by more basic results on so-called H-measures. These H-measures are quadratic default measures, introduced by Gérard [8] and Tartar [16]. They can be interpreted as two-point correlation functions of the underlying microstructure.

We have therefore rigorously derived the relaxed formulation of very general objective functions, including ones depending on the gradient of the state. Furthermore, due to the special “small amplitude” structure of the optimization problem we have devised efficient and simple numerical algorithms for computing the optimal shapes. These algorithms are gradient methods relying on the optimality conditions of the relaxed problem. A key ingredient is that optimal microstructures in small amplitude optimization can always be found in the class of simple or rank-one laminates. In other words, there are only two relevant design parameters in our method: the local volume fraction and the angle of lamination (which governs the anisotropy of the optimal microstructure). Another feature of our small amplitude method is that the coefficient of the state or adjoint equations are uniform and independent of the design. Indeed, all the geometric parameters appear as right hand sides in the equations. This implies a drastic reduction of the CPU cost of the method because, once the rigidity finite element matrix has been factorized by a Cholesky method, it is stored and used throughout the optimization process for different right hand sides at each iteration. We implemented our method only in two space dimensions using the FreeFem++ package for finite elements [10]. There is no conceptual difficulty in extending the method to three space dimensions where the gain in CPU time is even higher.

Of course, the small amplitude approximation is not really meaningful in the context of "standard" structural optimization which amounts to optimize the distribution of a given material with a very weak one mimicking holes (the so-called ersatz material approach). Indeed, the small amplitude assumption contradicts the fact that the ersatz component is much weaker than the reference one. However, it makes sense, for example, in the context of reinforced plane structures: a typical problem is to find the region where to reinforce the thickness of a plate by pasting some tape on top of it. Our method can be useful for this plane reinforcement problem and our numerical examples can be interpreted in this sense.

2. A model problem in conductivity

2.1 Small amplitude asymptotic

Let us consider mixtures of two conducting phases characterized by two symmetric positive definite tensors A^0 and A^1. We denote by η the amplitude or contrast or aspect ratio between the two materials. In other words, we assume that $A^1 = A^0(1+\eta)$. The range of η is restricted to $(-1;+\infty)$, but in the sequel we shall assume that η is a small parameter, i.e. $|\eta| << 1$. Denoting by χ the characteristic function of the region occupied by phase A^1, we define a conductivity tensor

$$A(x) = (1-\chi(x))A^0 + \chi(x)A^1 = A^0(1+\eta\chi(x)).$$

For a smooth bounded open set $\Omega \subset \mathbf{R}^N$, with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, and for given source terms $f \in H^{-1}(\Omega)$ and $g \in L^2(\partial\Omega)$, we consider the following boundary value problem

$$\left.\begin{array}{r} -\mathrm{div}\,(A\,\nabla u\,) = f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \Gamma_D \\ A\,\nabla u \cdot n = g \quad \text{on } \Gamma_N, \end{array}\right\} \qquad (1)$$

which admits a unique solution in $H^1(\Omega)$. Typically we want to minimize an objective function of the type

$$J(\chi) = \int_\Omega j_1(u)\,dx + \int_{\Gamma_N} j_2(u)\,ds,$$

where the boundary integral is defined only on Γ_N since $u = 0$ is fixed on Γ_D. We assume that the integrands j_i are of class C^3 with adequate growth conditions.

Assuming that the two phases have prescribed volume fractions, Θ for A^1 and $1-\Theta$ for A^0, with $\Theta \in (0,1)$, we define a set of admissible designs

$$\mathcal{U}_{ad} = \left\{ \chi \in L^\infty(\Omega;\{0,1\}), \text{ such that } \int_\Omega \chi(x)\,dx = \Theta|\Omega| \right\}. \tag{2}$$

We are ready to define the starting point of our study.

DEFINITION 1 *We call "large amplitude" optimal design problem the following optimization problem*

$$\inf_{\chi\in\mathcal{U}_{ad}} J(\chi). \tag{3}$$

Assuming that the amplitude or contrast η is small, we perform a second-order expansion in the state equation and in the objective function. Since the coefficient matrix A in (1) is an affine function of η, the solution $u \in H^1(\Omega)$ is analytic with respect to η, and we can write

$$u = u^0 + \eta\, u^1 + \eta^2 u^2 + O(\eta^3). \tag{4}$$

Plugging this ansatz in (1) yields three equations for (u^0, u^1, u^2)

$$\left.\begin{aligned} -\text{div}\,(A^0\,\nabla u^0\,) = f, \\ u^0 = 0 \quad \text{on } \Gamma_D \\ A^0\,\nabla u^0\cdot n = g \quad \text{on } \Gamma_N, \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} -\text{div}\,(A^0\,\nabla u^1\,) = \text{div}\,(\chi\, A^0 \nabla u^0\,), \\ u^1 = 0 \quad \text{on } \Gamma_D \\ A^0\,\nabla u^1\cdot n = -\chi A^0 \nabla u^0 \cdot n \quad \text{on } \Gamma_N, \end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} -\text{div}\,(A^0\,\nabla u^2\,) = \text{div}\,(\chi\, A^0 \nabla u^1\,) \\ u^2 = 0 \quad \text{on } \Gamma_D \\ A^0\,\nabla u^2\cdot n = -\chi A^0\,\nabla u^1 \cdot n \quad \text{on } \Gamma_N. \end{aligned}\right\} \tag{7}$$

Remark that u^0 does not depend on χ and thus only u^1, u^2 depends on χ. Similarly, we make a Taylor expansion in the objective function, and, neglecting the remainder term, we introduce a function $\mathcal{J}_{sa}$ which only depends on u^0, u^1, u^2

$$\begin{aligned} \mathcal{J}_{sa}(u^0,u^1,u^2) \quad &= \int_\Omega j_1(u^0)\,dx + \eta \int_\Omega j_1'(u^0)u^1\,dx \\ &+\eta^2 \int_\Omega \left(j_1'(u^0)u^2 + \frac{1}{2} j_1''(u^0)(u^1)^2 \right)\,dx \\ &+ \int_{\Gamma_N} j_2(u^0)\,ds + \eta \int_{\Gamma_N} j_2'(u^0)u^1\,ds \\ &+\eta^2 \int_{\Gamma_N} \left(j_2'(u^0)u^2 + \frac{1}{2} j_2''(u^0)(u^1)^2 \right)\,ds. \end{aligned} \tag{8}$$

DEFINITION 2 *We call "small amplitude" optimal design problem the second-order asymptotic of problem (3), namely*

$$\inf_{\chi\in\mathcal{U}_{ad}} \left\{ J_{sa}(\chi) = \mathcal{J}_{sa}(u^0, u^1, u^2) \right\} \tag{9}$$

where $\mathcal{J}_{sa}$ is defined by (8) and u^0, u^1, u^2 are solutions of the state equations (5), (6), (7) respectively.

2.2 Relaxation by H-measures

As most optimal design problems, the small amplitude problem (9) is ill-posed in the sense that it does not admit a minimizer in general. Therefore we relax it by using H-measure, a tool which was introduced by Gérard [8] and Tartar [16]. It is a default measure which allows to pass to the limit in quadratic functions of weakly converging sequences in $L^2(\mathbf{R}^N)$.

The general procedure for computing the relaxation of (9) is to consider a sequence (minimizing or not) of characteristic functions χ_n and to pass to the limit in (9) and its associated state equations. Up to a subsequence there exists a limit density θ such that χ_n converges weakly-* to θ in $L^\infty(\Omega;[0,1])$. We denote by u^0, u_n^1, u_n^2 the solutions of (5), (6), and (7) respectively, associated to χ_n (recall that (5) does not depend on χ_n). In a first step, it is easy to pass to the limit in the variational formulation of (6) to obtain that u_n^1 converges weakly to u^1 in $H^1(\Omega)$ which is the solution of

$$\left.\begin{array}{rcl} -\text{div}\,(A^0\,\nabla u^1\,) &=& \text{div}\,(\theta\, A^0 \nabla u^0\,) \quad \text{in} \;\; \Omega \\ u^1 &=& 0 \quad \text{on} \;\; \Gamma_D \\ A^0\,\nabla u^1 \cdot n &=& -\theta\, A^0 \nabla u^0 \cdot n \quad \text{on} \;\; \Gamma_N. \end{array}\right\} \tag{10}$$

The main difficulty comes from (7) where we need to pass to the limit in the product $\chi_n \nabla u_n^1$. Since this term is quadratic, we can use H-measures. More precisely, for a given test function $\phi \in H^1(\Omega)$ which vanishes on Γ_D, the variational formulation of (7) is

$$\int_\Omega A^0 \nabla u_n^2 \cdot \nabla\phi\, dx = -\int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla \phi\, dx. \tag{11}$$

The sequence u_n^2 is obviously bounded in $H^1(\Omega)$ and, up to a subsequence, it converges weakly to a limit u^2 in $H^1(\Omega)$. The question is to find which limit equation is satisfied by u^2. By using the theory of H-measures [16]), we deduce that

$$\begin{aligned} \lim_{n\to+\infty} \int_\Omega \chi_n A^0 \nabla u_n^1 \cdot \nabla\phi\, dx &= \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla \phi\, dx \\ &\quad - \int_\Omega \int_{\mathbf{S}_{N-1}} \theta(1-\theta) \frac{A^0 \nabla u^0 \cdot \xi}{A^0 \xi \cdot \xi} \xi \cdot A^0 \nabla \phi\, \nu(dx, d\xi), \end{aligned}$$

where ν is a probability measure with respect to ξ (see [2] for details). Introducing a matrix $M(x)$ defined by

$$M = \int_{\mathbf{S}^{N-1}} \frac{\xi \otimes \xi}{A^0 \xi \cdot \xi} \, \nu(x, d\xi) \tag{12}$$

we obtain that the limit of (11) is

$$\int_\Omega A^0 \nabla u^2 \cdot \nabla \phi \, dx = - \int_\Omega \theta A^0 \nabla u^1 \cdot \nabla \phi \, dx + \int_\Omega \theta(1-\theta) A^0 M A^0 \nabla u^0 \cdot \nabla \phi \, dx$$

for any smooth test function ϕ which vanishes on Γ_D. Thus u^2 is the unique solution in $H^1(\Omega)$ of

$$\left.\begin{array}{rcl} -\mathrm{div}\,(A^0 \, \nabla u^2\,) & = & \mathrm{div}\,(\theta \, A^0 \nabla u^1\,) - \mathrm{div}\,(\theta(1-\theta)\, A^0 M A^0 \nabla u^0\,) \quad \text{in } \Omega \\ u^2 & = & 0 \quad \text{on } \Gamma_D \\ A^0 \, \nabla u^2 \cdot n & = & -\theta \, A^0 \nabla u^1 \cdot n + \theta(1-\theta)\, A^0 M A^0 \nabla u^0 \cdot n \quad \text{on } \Gamma_N. \end{array}\right\} \tag{13}$$

We now can pass to the limit in the objective function $J_{sa}(\chi_n)$ to obtain

$$\lim_{n \to +\infty} J_{sa}(\chi_n) = J^*_{sa}(\theta, \nu) = \mathcal{J}_{sa}(u^0, u^1, u^2)$$

where u^0, u^1, u^2 are now solutions of the relaxed state equations (5), (10), (13), respectively. It is then a standard matter to prove the following result.

PROPOSITION 3 *The relaxation of (9) is thus*

$$\min_{(\theta,\nu) \in \mathcal{U}^*_{ad}} \left\{ J^*_{sa}(\theta, \nu) = \mathcal{J}_{sa}(u^0, u^1, u^2) \right\} \tag{14}$$

where $\mathcal{J}_{sa}(u^0, u^1, u^2)$ is defined by (8), u^0, u^1, u^2 are solutions of (5), (10), (13), respectively, and $\mathcal{U}^*_{ad}$ is defined by

$$\mathcal{U}^*_{ad} = \left\{ (\theta, \nu) \in L^\infty(\Omega; [0,1]) \times \mathcal{P}(\Omega, \mathbf{S}_{N-1}) \text{ s.t. } \int_\Omega \theta \, dx = \Theta |\Omega| \right\}, \tag{15}$$

where $\mathcal{P}(\Omega, \mathbf{S}_{N-1})$ is the set of probability measures on $\Omega \times \mathbf{S}_{N-1}$. More precisely, there exists at least one minimizer (θ, ν) of (14), any minimizer (θ, ν) of (14) is attained by a minimizing sequence χ_n of (9) in the sense that χ_n converges weakly- to θ in $L^\infty(\Omega)$, ν is the H-measure of $(\chi_n - \theta)$, and $\lim_{n\to+\infty} J_{sa}(\chi_n) = J^*_{sa}(\theta, \nu)$, any minimizing sequence χ_n of (9) converges in the previous sense to a minimizer (θ, ν) of (14).*

REMARK 4 *A simpler, albeit formal, method for computing the limits of u^1_n and u^2_n is to assume that the sequence χ_n of characteristic functions is periodically oscillating, i.e. $\chi_n(x) = \chi(x, nx)$ where $y \to \chi(x, y)$ is Y-periodic.*

Then, using formal two-scale asymptotic expansions it is possible to compute the limits of u_n^1 and u_n^2, as well as the first-order corrector term for u_n^1, i.e.

$$u_n^1(x) = u^1(x) + \frac{1}{n}u^{11}(x, nx) + O\left(\frac{1}{n^2}\right).$$

Making a Fourier expansion of $\chi(x,y) = \sum_{k\in\mathbf{Z}^N} \hat{\chi}(x,k)e^{2i\pi k\cdot y}$, we can compute explicitly u^{11} and u^2, and the H-measure is given by

$$\nu(x,\xi) = \frac{1}{\theta(1-\theta)} \sum_{k\neq 0\in\mathbf{Z}^N} |\hat{\chi}(x,k)|^2 \delta\left(\xi - \frac{k}{|k|}\right).$$

2.3 Optimality conditions

The goal of this section is to simplify the relaxed small amplitude optimization problem (14) by using information coming from its optimality conditions. The main result is that optimal microstructures for (14) can always be found in the class of simple laminates (i.e. rank-one laminates).

PROPOSITION 5 *The relaxed small amplitude problem (14) can be solved by restricting the set of probability measures $\mathcal{P}(\Omega, \mathbf{S}_{N-1})$ to its subset of Dirac masses. In other words, there exists an optimal design solution of (14) which is a simple laminate. Furthermore, the corresponding optimal H-measure, which is a Dirac mass, does not depend on the density θ.*

REMARK 6 *The main consequence of Proposition 5 is that not all possible composite materials have to be considered in the relaxed small amplitude problem (14) but just the simple laminates of rank one. It turns out that this property holds true for all generalizations of (14) [2]. Another interesting consequence of Proposition 5 is that the optimization with respect to ν can be done once and for all at the beginning of the optimization process since it is independent of the exact values of θ.*

Proof. To simplify the formula for $J_{sa}^*(\theta,\nu)$ which is implicit in ν, we introduce an adjoint state p^0 solution in $H^1(\Omega)$ of

$$\left.\begin{array}{rcl} -\text{div}\,(A^0\,\nabla p^0\,) & = & j_1'(u^0) \ \text{ in } \Omega \\ p^0 & = & 0 \ \text{ on } \Gamma_D \\ A^0\nabla p^0\cdot n & = & j_2'(u^0) \ \text{ on } \Gamma_N. \end{array}\right\} \tag{16}$$

Remark that, like u^0, the adjoint state p^0 does not depend on (θ,ν). The goal of this adjoint state is to eliminate u^2 in $J_{sa}^*(\theta,\nu)$. Indeed multiplying (16) by u^2 and integrating by parts, and doing the same for (13) multiplied by p^0, we obtain

$$
\begin{aligned}
J^*_{sa}(\theta,\nu) &= \int_\Omega j_1(u^0)\,dx + \eta\int_\Omega j_1'(u^0)u^1\,dx + \eta^2\int_\Omega \frac{1}{2}j_1''(u^0)(u^1)^2\,dx \\
&+ \int_{\Gamma_N} j_2(u^0)\,ds + \eta\int_{\Gamma_N} j_2'(u^0)u^1\,ds + \eta^2\int_{\Gamma_N}\frac{1}{2}j_2''(u^0)(u^1)^2\,ds \\
&-\eta^2\int_\Omega \theta A^0\nabla u^1\cdot\nabla p^0\,dx + \eta^2\int_\Omega \theta(1-\theta)A^0 M A^0\nabla u^0\cdot\nabla p^0\,dx
\end{aligned} \tag{17}
$$

which is now explicitly affine in M, defined by (12), and thus in ν since u^0 and p^0 are independent of ν and θ. Minimizing $J^*_{sa}(\theta,\nu)$ with respect to ν amounts to minimize a scalar affine function on the convex set of probability measures $\mathcal{P}(\Omega,\mathbf{S}_{N-1})$. Therefore any minimizer ν^* can be replaced by another minimizer which is a Dirac mass concentrated at a direction ξ^* which minimizes the integrand of the last term in (17). Remark that ξ^* does not depend on θ. Furthermore, replacing a minimizer ν^* by the Dirac mass concentrated at ξ^* does not change θ, u^0, u^1 and p^0. Thus one can restrict the minimization in ν to the subset of $\mathcal{P}(\Omega,\mathbf{S}_{N-1})$ made of Dirac masses of the type $\nu(x,\xi)=\delta(\xi-\xi^0(x))$. •

After elimination of the measure ν, i.e. incorporating the optimal Dirac mass concentrated on $\xi^*(x)$, we can further simplify the objective function by using again the adjoint p^0

$$
\begin{aligned}
J^*_{sa}(\theta) &= \int_\Omega j_1(u^0)\,dx + \int_{\Gamma_N} j_2(u^0)\,ds - \eta\int_\Omega \theta A^0\nabla u^0\cdot\nabla p^0\,dx \\
&+\frac{1}{2}\eta^2\int_\Omega j_1''(u^0)(u^1)^2\,dx + \frac{1}{2}\eta^2\int_{\Gamma_N} j_2''(u^0)(u^1)^2\,ds \\
&-\eta^2\int_\Omega \theta A^0\nabla u^1\cdot\nabla p^0\,dx + \eta^2\int_\Omega \theta(1-\theta)A^0 M^* A^0\nabla u^0\cdot\nabla p^0\,dx
\end{aligned}
$$

with $M^*=(\xi^*\otimes\xi^*)/(A^0\xi^*\cdot\xi^*)$. It is then a simple matter [2] to compute the derivative of J^*_{sa} with respect to θ.

LEMMA 7 *The objective function $J^*_{sa}(\theta)$ is Fréchet differentiable and its derivative in the direction $s\in L^\infty(\Omega)$ is given by*

$$
\begin{aligned}
\frac{\partial J^*_{sa}}{\partial\theta}(s) &= -\eta\int_\Omega sA^0\nabla u^0\cdot\nabla p^0\,dx - \eta^2\int_\Omega sA^0\nabla u^1\cdot\nabla p^0\,dx \\
&\quad -\eta^2\int_\Omega sA^0\nabla u^0\cdot\nabla p^1\,dx + \eta^2\int_\Omega s(1-2\theta)A^0M^*A^0\nabla u^0\cdot\nabla p^0\,dx,
\end{aligned}
$$

where p^1 is another adjoint state, defined as the solution in $H^1(\Omega)$ of

$$\left.\begin{array}{rcl} -\mathit{div}\,(A^0\,\nabla p^1\,) & = & j_1''(u^0)u^1 + \mathit{div}\,(\theta A^0\,\nabla p^0\,) \quad \mathit{in}\ \ \Omega \\ p^1 & = & 0 \quad \mathit{on}\ \ \Gamma_D \\ A^0\nabla p^1\cdot n & = & j_2''(u^0)u^1 - \theta A^0\nabla p^0\cdot n \quad \mathit{on}\ \ \Gamma_N. \end{array}\right\} \qquad (18)$$

2.4 Generalizations

The same method and the same results can be obtained for various other problems. For example, it is possible to derive the same results for an objective function that depends on the gradient of the state. We can also generalize our approach to the system of linearized elasticity by considering mixtures of two linear isotropic phases [2]. Furthermore, we can consider so-called multiple loads problems, i.e. several state equations associated to a single objective function. It is even possible to consider the case of a multi-physics problem, i.e. the coefficients of the different state equations can be different although they share the same geometry or microstructure (a typical example would be thermo-elasticity where a conductivity problem is coupled to an elasticity system). In all such cases, once again, simple laminates are optimal microstructures.

3. Algorithm and numerical examples

3.1 The optimization algorithm

We describe the optimization algorithm that we implemented to solve numerically the relaxed problems obtained in the previous sections. All the examples will be in dimension two. Recall that there are two design parameters: the lamination angle and the local proportion θ. We have proved that the lamination direction of the optimal microstructure does not depend on θ, and that it is explicitly given in terms of u^0 and p^0 which do not vary during the optimization process. Therefore, the optimal lamination angle is computed once and for all before we start a gradient-based steepest descend method for the local proportion θ.

The boundary value problems are solved using FreeFem++ [10] and we take advantage of the fact that all the problems we need to solve have the same elliptic differential operator, namely div $(A^0\nabla\ )$. Therefore the factorization of the stiffness matrix is performed only once during the initialization and is saved for all subsequent finite elements resolutions during the iterations. This of course speeds up considerably the code. The computational domain Ω is discretized by triangles. For all states u^i and adjoint states p^i we use P_2 Finite Elements, while the local proportion θ is discretized with P_0 Finite Elements (as well as the lamination direction ξ^*). As is well known (see [1], [5] and

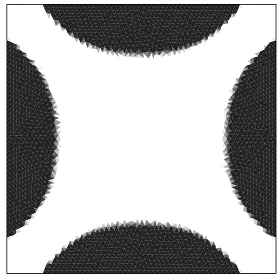

Figure 1. Gradient minimization. $\eta = -0.5$, Volume=40%.

references therein) we prefer the $P_2 - P_0$ combination to the simpler $P_1 - P_0$ in order to avoid the so-called checkerboard numerical instability.

The subsequent fi gures show the local proportion of the material with higher conductivity or with higher stiffness, meaning higher values of both Lamé parameters. In other words, if η is negative (which is always the case below), we display $(1 - \theta)$. The volume, when mentioned in the caption, always refers to the percentage of volume occupied by the better conductor or the stiffer material.

3.2 Diffusion Problem

Since the inception of the homogenization method a classical test case is the so-called torsion problem (see [1] for further references). It amounts to solve (1) in the unit square $\Omega = (0,1) \times (0,1)$ with Dirichlet boundary conditions and a source term $f \equiv 1$. We minimize $J_2(\chi) = \int_\Omega |\nabla u|^2 dx$. In Figure 1 we plot the resulting optimal shape for the relaxed small amplitude problem. The phase conductivities are 0.5 and 1, and the proportion of the best conductor is 40%. This resut is slightly different than that obtained by Lipton and Velo (see Figure 1:a in [11]) using a partial relaxation of the problem. Different values of η and different refi nement of the mesh yield similar results.

3.3 Elasticity Problem

In all the following examples we take the reference material A^0 with Lamé coeffi cients $\lambda = 0.73$ and $\mu = 0.376$. As we said in the introduction, one should interpret the following results in the context of reinforcing a plane structure by adding to it a layer at a location that is optimal.

Let us fi rst consider the so-called short cantilever problem subject to compliance minimization. We choose $\Omega = (0,1) \times (0,2)$ (discretized by 8765 triangles) clamped on its left side and with a unit vertical load at the middle of the right side. After 50 iterations the resulting optimal designs for $\eta = -0.1$

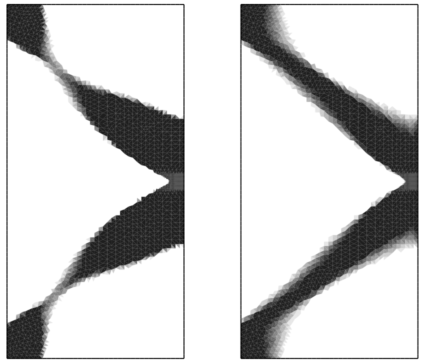

Figure 2. Compliance minimization for the short cantilever: $\eta = -0.1$ (left), $\eta = -0.99$ (right), volume=25%.

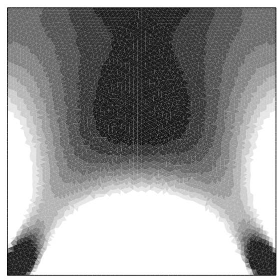

Figure 3. Strain minimization of a square clamped at the bottom and vertically loaded at the top: $\eta = -0.1$, volume=50%.

and $\eta = -0.99$ are shown on Figure 2 (recall that dark colors correspond to the stiffer material). The latter design is quite similar to the usual short cantilever with two bars making a 90 degree angle at the position where the load is applied, giving then the impression that the approach developed here for the small amplitude case, might very well be used at least in some cases when the amplitude is not necessarily so small.

Next we minimize the norm of the strain tensor, i.e. $J(\chi) = \int_\Omega |e(u)|^2 dx$. The domain is the unit square $\Omega = (0,1)^2$, which is discretized with 8654 triangles, clamped at the bottom and vertically loaded at the top. The resulting optimal design, shown in Figure 3, looks like a bridge with two pillars.

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References

[1] G. Allaire, *Shape Optimization by the Homogenization Method*. Springer-Verlag (2002).

[2] G. Allaire, S. Gutierrez, *Optimal Design in Small Amplitude Homogenization,* submitted. CMAP preprint 576 (2005).

[3] G. Allaire, F. Jouve, H. Maillot, *Topology Optimization for Minimum Stress Design with the Homogenization Method*. Struct. Multidiscip. Optim. 28, no. 2-3, 87–98 (2004).

[4] J.C. Bellido, P. Pedregal, *Explicit quasiconvexifi cation for some cost functionals depending on derivatives of the state in optimal designing,* Discr. Contin. Dyn. Syst. 8, no. 4, 967–982 (2002).

[5] M. Bendsoe, O. Sigmund, *Topology Optimization. Theory, Methods, and Applications*. Springer-Verlag, New York (2003).

[6] A. Cherkaev, *Variational Methods for Structural Optimization,* Springer Verlag, New York (2000).

[7] P. Duysinx, M. Bendsoe, Topology Optimization of Continuum Structures with Local Stress Constraints, *Int. J. Num. Meth. Engng.*, 43, pp.1453-1478 (1998).

[8] P. Gerard, *Microlocal defect measures,* Comm. Partial Diff. Equations 16, pp.1761-1794 (1991).

[9] Y. Grabovsky, Optimal design problems for two-phase conducting composites with weakly discontinuous objective functionals, *Adv. in Appl. Math.*, 27, pp.683-704 (2001).

[10] F. Hecht, O. Pironneau, K. Ohtsuka, *FreeFem++ Manual,* downloadable at `http://www.freefem.org`

[11] R. Lipton, A. Velo, *Optimal design of gradient fi elds with applications to electrostatics*. Nonlinear partial differential equations and their applications. College de France Seminar, Vol. XIV 509-532. Stud. Math. Appl., 31 (2002).

[12] R. Lipton, *Relaxation through homogenization for optimal design problems with gradient constraints*. J. Optim. Theory Appl. 114, no. 1, 27–53 (2002).

[13] R. Lipton, *Stress constrained G closure and relaxation of structural design problems*. Quart. Appl. Math. 62, no. 2, 295–321 (2004).

[14] G. Milton, *The theory of composites,* Cambridge University Press (2001).

[15] F. Murat, L. Tartar, *Calcul des Variations et Homogénéisation,* Les Méthodes de l'Homogénéisation Théorie et Applications en Physique, Coll. Dir. Etudes et Recherches EDF, 57, Eyrolles, Paris, pp.319-369 (1985). English translation in *Topics in the mathematical modelling of composite materials,* A. Cherkaev and R. Kohn editors, Progress in Nonlinear Differential Equations and their Applications, 31, Birkhäuser, Boston (1997).

[16] L. Tartar, *H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations*, Proc. of the Royal Soc. Edinburgh, 115A, pp. 193–230 (1990).

[17] L. Tartar, *Remarks on optimal design problems*. Calculus of variations, homogenization and continuum mechanics (Marseille, 1993), 279–296, Ser. Adv. Math. Appl. Sci., 18, World Sci. Publishing, River Edge, NJ (1994).

[18] L. Tartar, *An introduction to the homogenization method in optimal design,* in Optimal shape design (Tróia, 1998), A. Cellina and A. Ornelas eds., Lecture Notes in Mathematics 1740, pp.47-156, Springer, Berlin (2000).