A NUMERICAL ALGORITHM FOR TOPOLOGY AND SHAPE OPTIMIZATION

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ABSTRACT. In the context of topology and shape optimization, we minimize the sum of the elastic compliance and of the weight of a two-dimensional structure under specified loading. A relaxed formulation of the original problem which uses composites obtained by microperforation is introduced. A new numerical algorithm is proposed ; it provides a natural link between the previously known method of Bendsoe, Kikuchi, and Suzuki, and that of Allaire and Kohn.

1. Motivation

It is by now well established that the shape optimization problem which consists in the minimization of the sum of the compliance and of the weight for an elastic body under a given load does not necessarily possess a solution among regular domains. A generalized solution may however be obtained upon allowing for microperforations. The resulting relaxed problem is obtained by homogenization of these microstructures, and is described in [1], [2] (for an introduction to the use of homogenization for relaxation in variational problems, see [9], [10], [11], and the references therein). It will be assumed that the reader is familiar with the shape optimization problem described in the paper [1] in this volume (for further insight into the problem see also the list of references in [2]).

Our goal in this study is to take full advantage of the intimate knowledge one has about the relaxed formulation to present an efficient numerical algorithm. Specifically we will rely on the optimal character of a certain class of microstructures, namely laminates.

The relaxed formulation described in [1], [2] reads as follows :

$$\text{Min}_{\substack{div\ \tau = 0\ in\ \Omega \\ \tau.n = f\ on\ \partial\Omega}} \int_\Omega \text{Min}_{\substack{0 \le \theta \le 1 \\ A \in G_\theta}} \left[< A^{-1}\tau,\tau > + \lambda\,\theta \right] dx, \tag{1.1}$$

where f is a given boundary loading, θ is the volume fraction of material in the composite, G_θ is the set of all possible effective elasticity tensors for a given density θ, and λ is a positive Lagrange multiplier whose effect is to modulate the relative contributions of the compliance and of the weight.

In [2] the minimization of $<A^{-1}\tau,\tau>$ over G_θ (the theoretically delicate step) is performed with the help of the Hashin-Shtrikman variational principle. But it is also known [3] that, for a

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given τ, this minimum is attained by an effective tensor A corresponding to a rank-N lamination in N space dimensions. The reader is referred to [6] for a detailed presentation of multiple layering in the context of elasticity, and to [3] for the optimality argument that leads to the aforementioned result. Further, according to [8] and [3], the directions of lamination coincide with the eigendirections of τ. Note that this last feature of the optimal composite is specific to the problem of minimization of $<A^{-1}\tau,\tau>$; it does not hold true in the case where $<A\varepsilon,\varepsilon>$ is the quantity to minimize, for a given strain ε (cf. [3], [7]). It is at the root of the algorithm presented in this paper.

Thus far two numerical methods have been implemented, that proposed by Bendsoe, Kikuchi, and Suzuki [5], [12], and that of Allaire and Kohn [2]. The former transforms the minimization over statically admissible stresses τ into a maximization over displacements, and (1.1) becomes

$$\text{Max}_{\substack{0\le \theta(x) \le 1 \\ A(x)\in G_{\theta(x)}}} \text{Min}_{u\in H^1(\Omega;\mathbb{R}^N)}\left\{\int_\Omega\left[\frac{1}{2}< A(x)e(u),e(u)> + \lambda\,\theta(x)\right] dx - \int_{\partial\Omega}f.u\right\}. \qquad (1.2)$$

It then specializes the minimum over $G_{\theta(x)}$ to a minimum over rank-N laminates (at least in the version of this method presented in [4]), and proposes a numerical scheme based on the first order optimality conditions at the saddle point of the functional (1.2). This leads to a rather intricate updating process for the design variables (the volume fraction of material θ, as well as the individual volume fractions and orientations of each layer). The computation is performed using "alternate directions" ; firstly the solution u of a linear elasticity problem where all design variables are fixed is obtained, then the design variables are updated using the optimality criterion.

The latter approach performs analytically the minimization over θ and G_θ for each τ, and thus (1.1) becomes

$$\text{Min}_{\substack{div\ \tau = 0\ in\ \Omega \\ \tau.n = f\ on\ \partial\Omega}} \int_\Omega F_\lambda(\tau)\,dx\ , \qquad (1.3)$$

where F_λ is a highly non linear and non-convex function of τ (see [1]). The design variable θ is easily recovered from the minimizer τ in (1.3) through algebraic optimality conditions for θ. From a numerical standpoint, a two-dimensional non-linear elasticity problem is to be solved ; the energy F_λ is then minimized with an iterative conjugate gradient method. As in all computations involving complementary energies, high degree finite elements have to be used to accurately compute the stress. For this reason, as well as for lack of a completely explicit expression of F_λ, this approach is restricted, for all practical purpose, to the two-dimensional setting.

We outline in the next section an algorithm which sits right in between the above described methods. It does not use the strain formulation which would lead, as in [5], [12], to a min-max problem, but it remains at the stress formulation level for which it is a double minimization problem. At the same time, it avoids the solving of a non-linear non-convex minimization problem in τ as in [2]. It is based on an alternate direction method which consists, as in [5], [12], in solving, for a given set of design variables, the associated linear elasticity problem (and to that extent could be performed in a strain as well as in a stress based setting), then in an update of the design variables. In contrast with [5], [12], that update is however straightforward because the available information about the optimal character of multiple layers is put to full use. Specifically, for a given stress τ, we use explicit formulae available for all design variables of the associated optimal layered microstructure.

The advantages of this algorithm are manyfold. It is a full minimization problem ; no non-linear problems are solved ; the update of the design variables is explicit and optimal at the same time ; its extension to three space dimensions is practically feasible. Our so far brief numerical experience with this algorithm shows fast speed of convergence, although admittedly at the expense of a penalization of extremely low volume fractions. Note that both previous methods have to resort to some kind of penalization of the very low volume fraction range.

2. The Design Variables Update

From now onward the setting is two-dimensional. Let us assume that the stress field τ at a given iteration is known. It is then a direct consequence of the results of [2] that an optimal microstructure consists in a rank-two layering of the material with void (physically, it looks like very thin and long holes in a matrix of material). The directions of lamination coincide with the eigendirections e_1, e_2 of τ, corresponding to its eigenvalues τ_1, τ_2. For each successive lamination, the volume fraction of material is $m_1\theta$, $m_2\theta$ respectively, where θ is the overall volume fraction of material and $m_1 + m_2 = 1$.

The material under consideration is assumed to be isotropic with bulk and shear moduli κ and μ. In other words, for any 2×2 symmetric matrix ε, its elasticity tensor A_0 is defined by

$$A_0 \varepsilon = \kappa \, (tr \, \varepsilon) \, I \; + \; 2\mu \left[\varepsilon - \frac{tr \, \varepsilon}{2} I \right], \tag{2.1}$$

where I is the identity matrix.

We consider a rank-two layering of the material A_0 with void, in the directions e_1, e_2, and with volume fractions $m_1\theta$, $m_2\theta$. For any stress τ, whose principal axes (or eigendirections) coincide with the layering directions, the associated effective elasticity tensor A is such that

$$< A^{-1}\tau,\tau > \; = \; < A_0^{-1}\tau,\tau > \; + \; \frac{(\kappa+\mu)(1-\theta)}{4\kappa\mu\theta} \left[\frac{\tau_1^2}{m_2} + \frac{\tau_2^2}{m_1} \right]. \tag{2.2}$$

The above formula will not be established in detail here. It immediately results from a more general formula which yields the effective tensor A (or rather its inverse) of a rank-N layering of two isotropic materials (see formula (6.11) in [3] or proposition 4.2 in [6]).

It thus remains to minimize (2.2) with respect to m_1, m_2, with the constraint $m_1 + m_2 = 1$. We obtain

$$m_1 = \frac{|\tau_2|}{|\tau_1| + |\tau_2|} \; , \quad m_2 = \frac{|\tau_1|}{|\tau_1| + |\tau_2|}. \tag{2.3}$$

Thus, we have

$$\text{Min}_{m_1, m_2} < A^{-1}\tau,\tau > \; = \; < A_0^{-1}\tau,\tau > \; + \; \frac{(\kappa+\mu)(1-\theta)}{4\kappa\mu\theta} \left[|\tau_1| + |\tau_2| \right]^2. \tag{2.4}$$

The computation of the optimal θ becomes obvious. It must minimize, over the interval [0;1], the quantity

$$\frac{(\kappa+\mu)(1-\theta)}{4\kappa\mu\theta)} \left[|\tau_1| + |\tau_2| \right]^2 \; + \; \lambda \, \theta. \tag{2.5}$$

This yields

$$\theta = \left[\frac{(\kappa + \mu)}{4 \kappa \mu \lambda} \right]^{1/2} \left[|\tau_1| + |\tau_2| \right] \tag{2.6}$$

if this quantity is less than 1, and $\theta = 1$ otherwise.

We now have at our disposal all the theoretical ingredients for the optimization algorithm. We fix a domain Ω, a boundary loading f, and a Lagrange multiplier λ; the algorithm reads as :

initialization : set $\theta = 1$ everywhere, i.e. start from the design coinciding with the domain Ω full of material.

iterations : until convergence, compute the linear elasticity problem with the Hooke's law A corresponding to the previous design, then, from the solution stress τ, deduce the new design variables θ, m_i, e_i, and Hooke's law A.

In practice, the finite element code used to solve the elasticity problem is that of [1]. Thus, the stress field is computed as the second derivatives of the Airy potential. The stopping criterion in the above algorithm is the L^2-norm of the difference between the two last vectors of degrees of freedom for the discretized Airy potential. The design variables θ, m_i, and e_i are computed at the Gauss points that are used for integrating the Airy potential.

There is a subtle point in using rank-two layered microstructures in two dimensions : they do not support stresses whose eigendirections do not coincide with the layering directions. This fact is peculiar to 2-D ; it does not happen in 3-D for rank-three layerings. However, this is not a major difficulty, since at the optimum (i.e. when the above algorithm converges) the stress and the layered microstructure have the same eigendirections. Before convergence it suffices to add to the elastic energy $< A\varepsilon, \varepsilon >$ a term of the form $(\varepsilon e_1 . e_2)^2$.

3. Numerical Results

We present two computations, and we briefly describe the accompanying results. For both cases, the Lamé moduli are set to $\kappa = 1.0$ and $\mu = 0.5$, and the Lagrange multiplier λ is equal to 1. To avoid very low volume fraction (i.e. degeneracy of the effective elasticity tensor A), the lowest possible values of θ, m_1, and m_2 are fixed at 10^{-3}.

Firstly, the fillet problem is investigated on a square domain meshed with 128 triangles (with 3 degrees of freedom per triangle). It is submitted to a uniformly distributed traction along one side, and to a uniformly distributed traction along the middle half of the opposite side. The net forces are balanced so as to ensure equilibrium. We have run 50 iterations, and figures 1 and 2 shows the convergence of the total energy (the sum of the compliance and of the weight) and of the renormalized weight (i.e. divided by the weight at the first iteration). The density of the resulting design is shown on figure 3 (white is void and black is pure material) : almost 53 % of material has been removed from the original square domain.

Secondly, the beam problem is investigated on a rectangular domain with length fourth times its width. By symmetry, only half of the beam is meshed with 256 triangles (with 3 degrees of freedom per triangle). It is submitted to a uniformly distributed traction along the middle 1/16 of its upper face, and it is clamped symmetrically on the left and right 1/32 of its lower side. We have run 100 iterations, and figures 4 and 5 shows the convergence of the
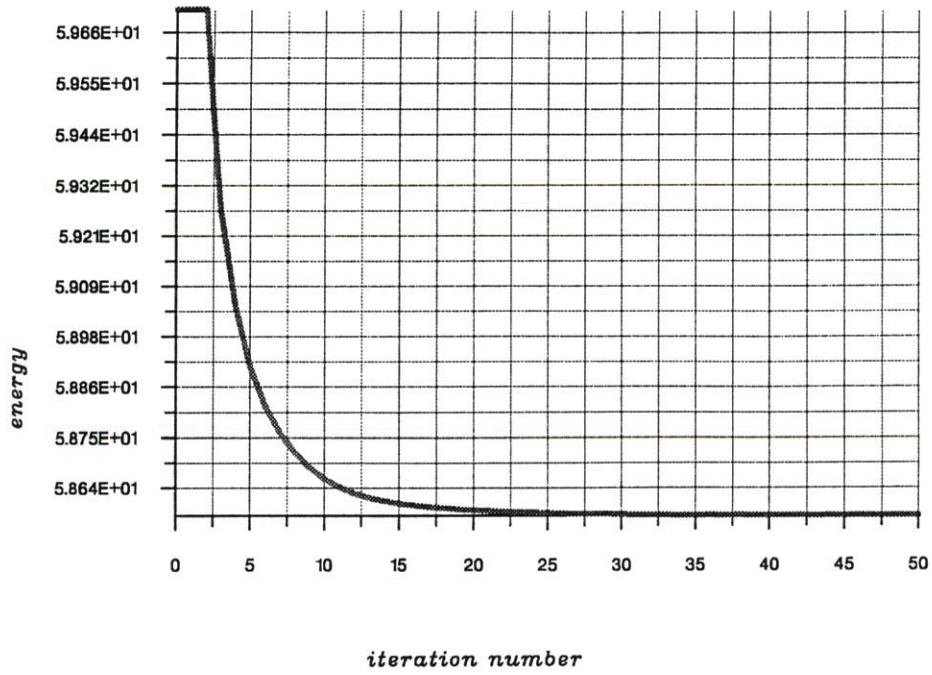
total energy and of the renormalized weight. The density of the resulting design is shown on figure 6 : almost 60 % of material has been removed from the original rectangular domain.

Remark that the results of both cases are very similar to those of [1], [2]. In particular they include large regions of composite material, in contrast with the results obtained in [5], [12]. Nevertheless, we can "penalize", as in [1], the use of composites by changing slightly formula (2.6) to force θ to be close to 0 or 1. We have tried this penalization procedure on the beam problem. Starting from the solution shown on figure 6, we obtained figure 7 which is closer to the type of results obtained in [5], [12].

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Figure 1
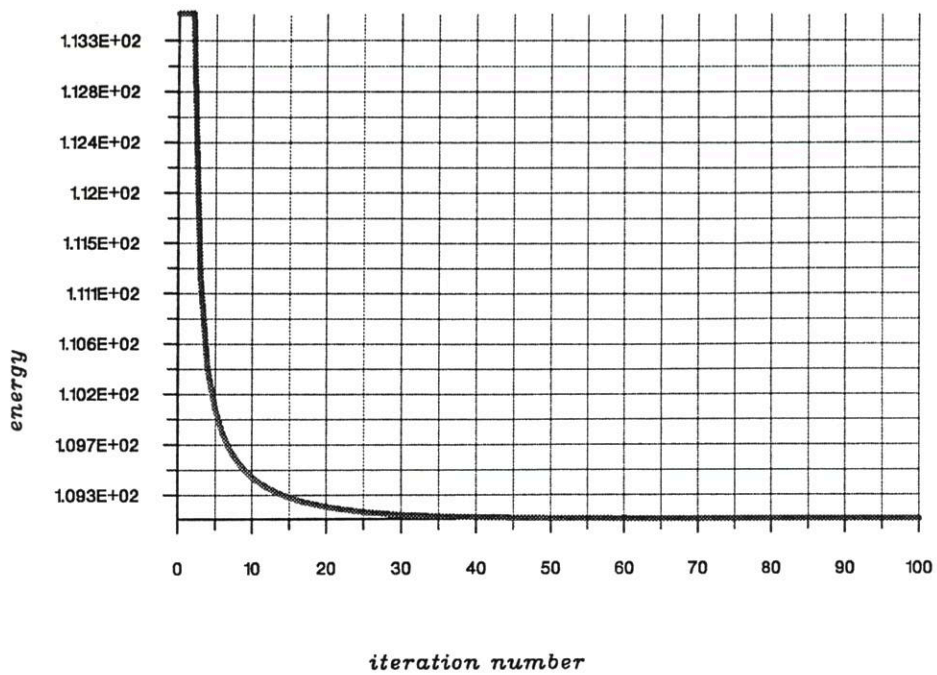fillet : convergence history for the energy
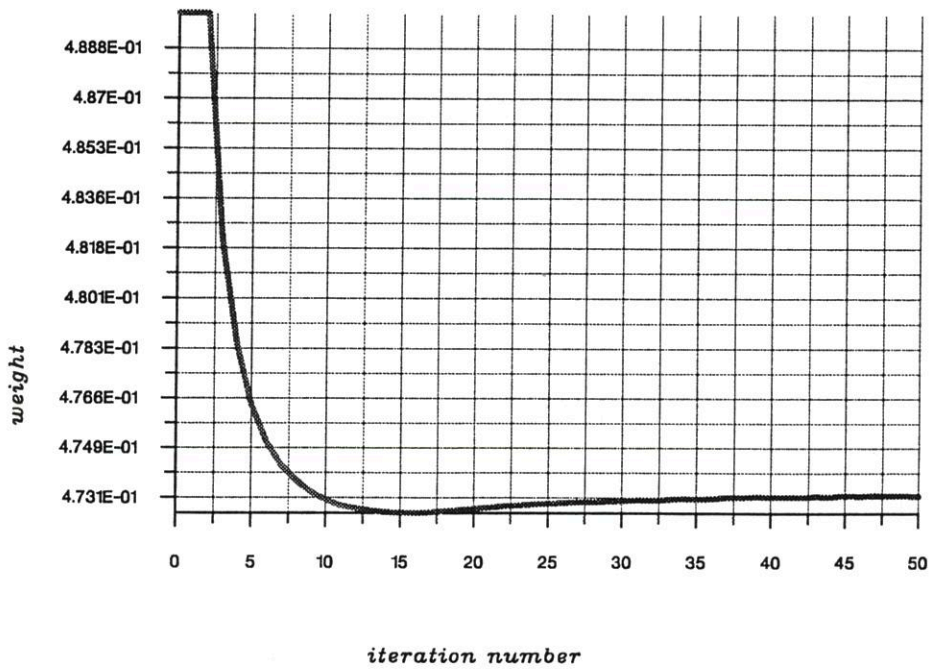
Figure 2
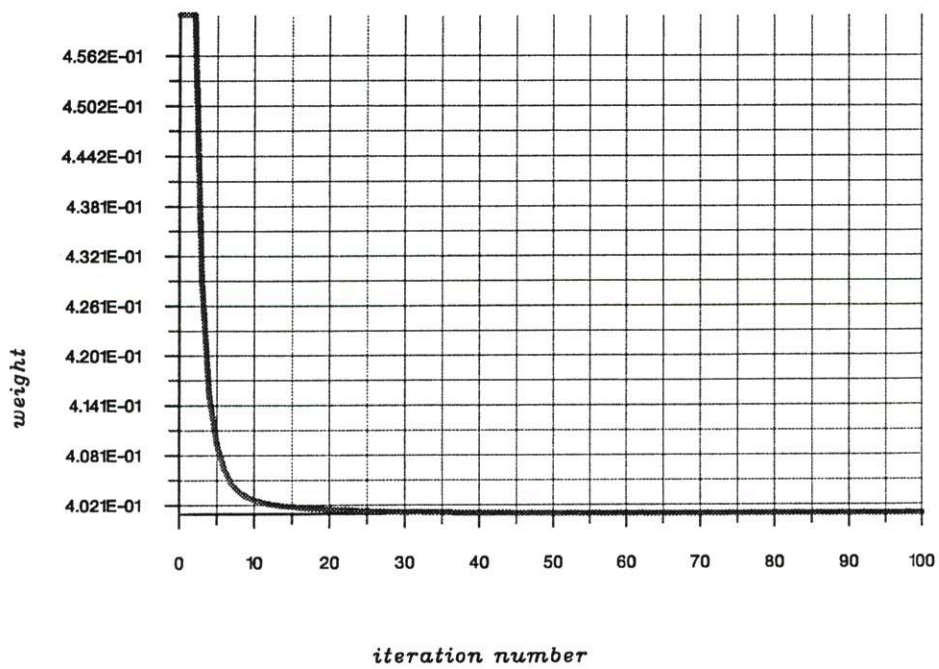fillet : convergence history for the renormalized weight
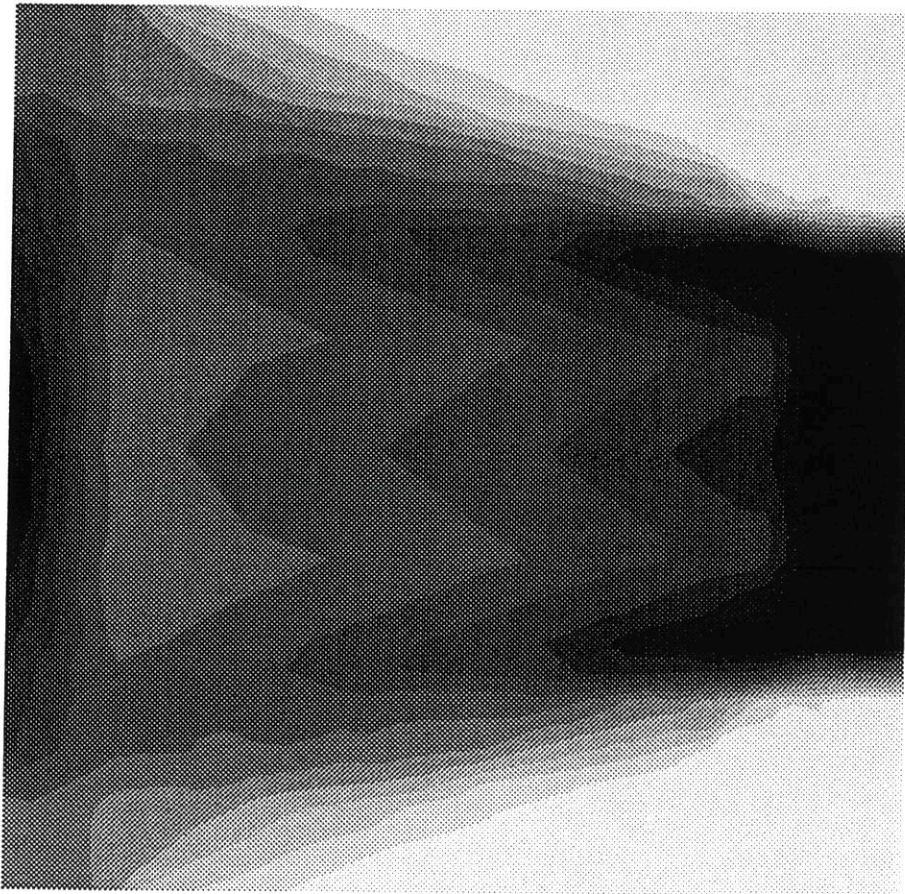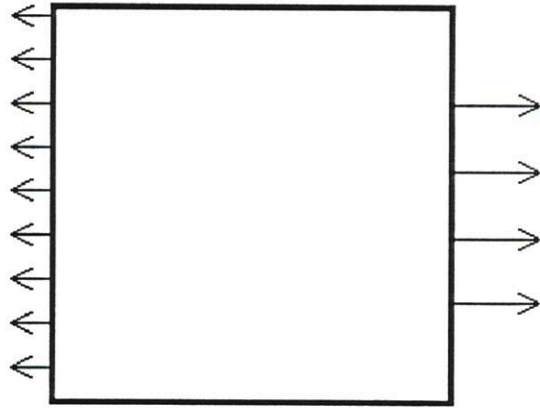
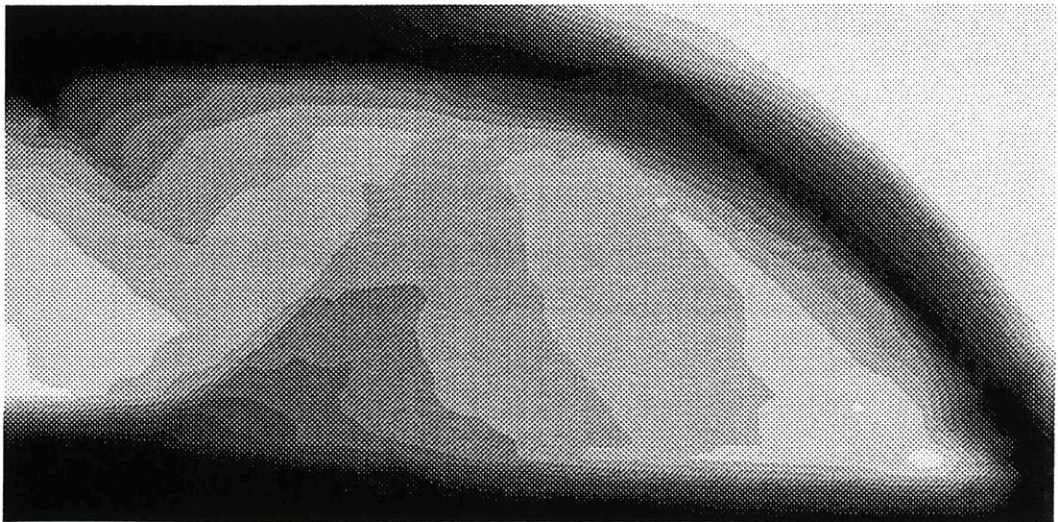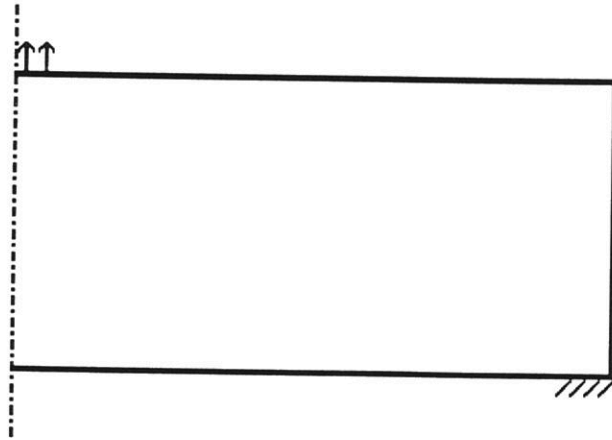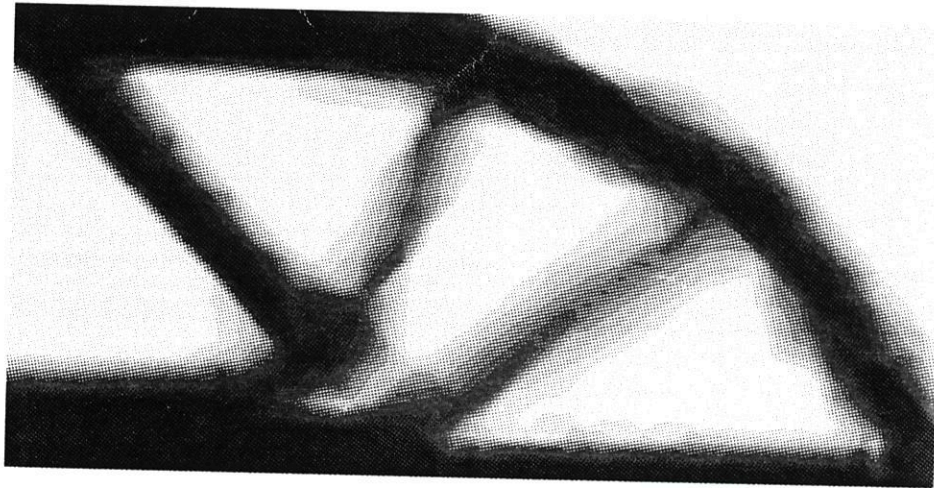**Figure 3**
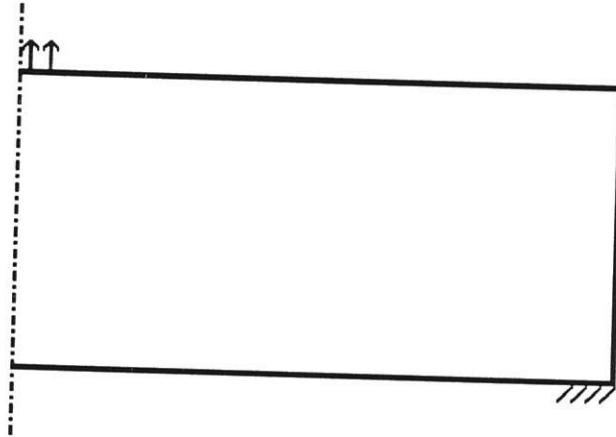fillet : density of the optimal design

Figure 4
beam : convergence history for the energy

Figure 5
beam : convergence history for the renormalized weight

Figure 6
beam : density of the optimal design

Figure 7
beam : density of the design where composite is penalized