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BLOCH WAVE HOMOGENIZATION AND SPECTRAL ASYMPTOTIC ANALYSIS

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ABSTRACT. – We consider a second-order elliptic equation in a bounded periodic heterogeneous medium and study the asymptotic behavior of its spectrum, as the structure period goes to zero. We use a new method of *Bloch wave homogenization* which, unlike the classical homogenization method, characterizes a renormalized limit of the spectrum, namely sequences of eigenvalues of the order of the square of the medium period. We prove that such a renormalized limit spectrum is made of two parts: the so-called *Bloch spectrum*, which is explicitly defined as the spectrum of a family of limit problems, and the so-called *boundary layer spectrum*, which is made of limit eigenvalues corresponding to sequences of eigenvectors concentrating on the boundary of the domain. This analysis relies also on a notion of *Bloch measures* which can be seen as ad hoc Wigner measures in the context of semi-classical analysis. Finally, for rectangular domains made of entire periodicity cells, a variant of the Bloch wave homogenization method gives an explicit characterization of the boundary layer spectrum too. © Elsevier, Paris

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RÉSUMÉ. – On considère une équation elliptique du deuxième ordre dans un milieu périodique hétérogène borné, et on étudie le comportement asymptotique de son spectre lorsque la période tend vers zéro. On utilise une nouvelle méthode d'*homogénéisation par ondes de Bloch* qui, contrairement aux méthodes classiques d'homogénéisation, caractérise la limite renormalisée du spectre, et plus précisément les suites de valeurs propres de l'ordre du carré de la période. On démontre que le spectre limite renormalisé est constitué de deux parties : un *spectre de Bloch*, qui est explicitement caractérisé comme le spectre d'une famille de problèmes limites, et un *spectre de couche limite*, qui est l'ensemble des limites de suites de valeurs propres dont les vecteurs propres correspondants se concentrent sur le bord du domaine. L'analyse présentée repose sur une notion de *mesures de Bloch* qui peuvent être vues comme des versions ad hoc des mesures de Wigner utilisées en analyse semi-classique. Enfin, pour des domaines rectangulaires constitués uniquement de cellules de périodicité entières, une variante de la méthode d'homogénéisation par ondes de Bloch permet de donner aussi une caractérisation explicite du spectre de couche limite. © Elsevier, Paris

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1. Introduction

Given a smooth bounded domain Ω in $\mathbb{R}^N$ occupied by a periodic heterogeneous medium, of period $\epsilon \in \mathbb{R}^+$, we consider the following spectral problem for the wave equation in Ω. Find all couples $(\lambda_\epsilon, v_\epsilon) \in \mathbb{R}^+ \times H_0^1(\Omega), v_\epsilon \not\equiv 0$, such that

$$\text{(1)} \qquad \begin{cases} -\operatorname{div}\left[A\left(x, \frac{x}{\epsilon}\right)\nabla v_\epsilon\right] = \frac{1}{\lambda_\epsilon} v_\epsilon & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega\ . \end{cases}$$

The coefficients of equation (1) are given by a coercive, symmetric matrix $A(x, y)$ which is smooth as a function of x and Y-periodic as a function of y (Y denotes the unit cube $[0,1]^N$). More precisely, we assume that

$$\text{(2)} \qquad A(x, y) \in \mathcal{C}(\overline{\Omega}; L^\infty_\#(Y)^{N\times N}).$$

In particular, assumption (2) implies that $A\left(x, \frac{x}{\epsilon}\right)$ is a measurable function in $L^\infty(\Omega)^{N\times N}$ and the spectral problem (1) is well-posed. Let σ_ϵ be the spectrum of (1), *i.e.* the set of eigenvalues λ_ϵ solutions of (1). As is well-known, for fixed ϵ, and since Ω is bounded, the spectrum σ_ϵ is discrete, made of a countable sequence of eigenvalues converging to 0 (plus the accumulation point 0)

$$\text{(3)} \qquad \sigma_\epsilon = \{0\} \cup \bigcup_{k\geq 1} \{\lambda_\epsilon^k\} \quad \text{with} \quad \lambda_\epsilon^1 \geq \lambda_\epsilon^2 \geq \cdots \geq \lambda_\epsilon^k \geq \cdots \to 0.$$

To each eigenvalue λ_ϵ^k is associated a normalized eigenfunction $v_\epsilon^k \in L^2(\Omega)$ such that $\|v_\epsilon^k\|_{L^2(\Omega)} = 1$ and the family $\{v_\epsilon^k\}_{k\geq 1}$ is an orthonormal basis of $L^2(\Omega)$.

The purpose of our work is to study the asymptotic behavior of the spectrum σ_ϵ when the period ϵ goes to 0. The second-order elliptic partial differential equation (1) is just a model problem. Our original motivation comes from more complicated models, describing the vibrations of fluid-solid structures, which were introduced by Planchard [39], [40] and extensively studied in [1], [16], [17], [18]. Actually, all our new results presented here were first applied to this problem of fluid-solid structures in [6], [7]. Our goal here is to expose in a single self-contained paper our complete theory in a systematic way on a simpler problem. Other motivations for studying the asymptotic behavior of σ_ϵ are the numerical computation of solutions of the wave equation in periodic media (*cf.* [33], [45], [46]), and the control of the wave equation in such media (*cf.* [15]). Let us emphasize that our method for studying the limit behavior of σ_ϵ works equally well for a vector or a scalar equation, and is indifferent to the type of boundary conditions.

In the next section we shall recall classical results of homogenization which describe completely the "usual" limit of σ_ϵ by finding the limit of each eigenvalue λ_ϵ^k when ϵ goes to 0 with fixed index k. Such a limit is usually called a *low frequency limit*. Indeed, recall that the vibration eigenfrequencies for the wave equation are related to the eigenvalues of (1) by

$$(\omega_\epsilon^k)^2 = \frac{1}{\lambda_\epsilon^k}.$$

Physically, it means that the low frequency limit gives the homogenized behavior of eigenmodes which vary on a scale much larger than the period ϵ. This situation is by now fairly well understood. However, a physically relevant case is the so-called *high frequency limit*, *i.e.* the asymptotic behavior of eigenvalues λ_ϵ^k which converge to 0 when ϵ goes to 0 and k to $+\infty$.

In section 3 we state our main new results concerning this high frequency limit of σ_ϵ. We characterize renormalized limits of the type $\lim_{\epsilon\to 0} a_\epsilon^{-2}\sigma_\epsilon$ where a_ϵ is an eigenfrequency scaling which goes to 0 with ϵ. For all scalings a_ϵ such that either $a_\epsilon << \epsilon$ or $a_\epsilon >> \epsilon$, we show that the limit of $a_\epsilon^{-2}\sigma_\epsilon$ is simply the entire half-line $\mathbb{R}^+$. When $a_\epsilon = \epsilon$, we prove a deeper result, namely that the limit of $\epsilon^{-2}\sigma_\epsilon$ is made of two parts: the so-called Bloch spectrum, which is explicitly defined as the spectrum of a family of limit problems, and the so-called boundary layer spectrum, which is made of limit eigenvalues corresponding to sequences of eigenvectors concentrating on the boundary of the domain. We refer to section 3 for a more detailed discussion of our results.

In section 4 we apply our new method of Bloch wave homogenization to equation (1) in order to prove that the Bloch spectrum is indeed part of the limit of $\epsilon^{-2}\sigma_\epsilon$. The Bloch wave homogenization method is a combination of two-scale convergence (*see* [2], [36]) and of Bloch wave decomposition (also known as Floquet decomposition, *see* [12], [22]).

In section 5 we prove a completeness result which states that the difference between the limit of $\epsilon^{-2}\sigma_\epsilon$ and the Bloch spectrum is exactly equal to the boundary layer spectrum. Our main tool is the notion of Bloch measures which is a new type of default measure, very similar to the Wigner measure (*see* [24], [31], [32]) although specific to the present situation.

In section 6 we prove that all other renormalized limits of $a_\epsilon^{-2}\sigma_\epsilon$ with either $a_\epsilon << \epsilon$ or $a_\epsilon >> \epsilon$ are equal to $\mathbb{R}^+$. In such a case, there is no interaction of the singular perturbation at scale a_ϵ and the homogenization at scale ϵ, and this result is obtained by using the notion of three-scale convergence.

Finally section 7 is devoted to a complete study of the boundary layer spectrum when the domain Ω and the sequence ϵ are chosen in such a way that Ω is always the union of a finite number of entire periodicity cells. In this case, the boundary layer spectrum is explicitly characterized as the spectrum of a new family of limit problems associated to the boundary of Ω.

2. Classical homogenization

The question of finding the limit of the spectrum σ_ϵ has already attracted a lot of attention. Indeed, using the classical homogenization technique (as described, *e.g.* in [9], [10], [27], [35], [44]), the low-frequency or homogenized limit of σ_ϵ has been found in [13], [29], [38], [48]. We briefly describe the procedure to obtain this homogenized limit.

As is well known, for fixed ϵ, to find the spectrum, *i.e.* the set of all solutions $(\lambda_\epsilon, v_\epsilon) \in \mathbb{R}^+ \times H_0^1(\Omega), v_\epsilon \not\equiv 0$, of

$$\begin{cases} -\operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla v_\epsilon\right] = \frac{1}{\lambda_\epsilon} v_\epsilon & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

is equivalent to the spectral study of the following linear operator

$$(4) \qquad \begin{cases} S_\epsilon : L^2(\Omega) & \longrightarrow L^2(\Omega) \\ \qquad f & \longrightarrow u_\epsilon, \end{cases}$$

where u_ϵ is the unique solution in $H_0^1(\Omega)$ of

$$(5) \qquad \begin{cases} -\operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla u_\epsilon\right] = f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

It is easily seen that S_ϵ is a self-adjoint compact operator in $\mathcal{L}(L^2(\Omega))$. Its spectrum is discrete made of a countable sequence of eigenvalues converging to 0 (plus the limit point 0)

$$\sigma(S_\epsilon) = \{0\} \cup \bigcup_{k\geq 1}\{\lambda_\epsilon^k\} \quad \text{with} \quad \lambda_\epsilon^1 \geq \lambda_\epsilon^2 \geq \cdots \geq \lambda_\epsilon^k \geq \cdots \to 0.$$

To each λ_ϵ^k is associated a normalized eigenfunction $v_\epsilon^k \in L^2(\Omega)$ such that $\|v_\epsilon^k\|_{L^2(\Omega)} = 1$ and the family $\{v_\epsilon^k\}_k$ is an orthonormal basis of $L^2(\Omega)$.

To describe the limit or homogenized operator, we introduce the homogenized equation for (5). Let us define first a homogenized matrix $A^*(x)$, for almost any $x \in \Omega$, by

$$(6) \qquad A^*(x)\zeta \cdot \zeta = \min_{\phi \in H^1_\#(Y)} \int_Y A(x,y)(\zeta + \nabla\phi(y)) \cdot (\zeta + \nabla\phi(y))dy \quad \forall \zeta \in \mathbb{R}^N.$$

REMARK 2.1. – *Since $A(x, y)$ is symmetric by definition, formula* (6) *makes sense and defines a unique symmetric matrix $A^*(x)$. Furthermore, $A^*(x)$ enjoys the same coercivity and boundedness properties than $A(x, y)$.*

Then, a limit operator S is defined by

$$\begin{cases} S : L^2(\Omega) & \longrightarrow L^2(\Omega) \\ \qquad f & \longrightarrow u, \end{cases}$$

where u is the unique solution in $H_0^1(\Omega)$ of the homogenized equation for (5)

$$\begin{cases} -\operatorname{div}[A^*(x)\nabla u] = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Clearly S is a self-adjoint compact operator in $\mathcal{L}(L^2(\Omega))$. Its spectrum $\sigma(S)$ is exactly

$$\sigma(S) = \{0\} \cup \bigcup_{k\geq 1} \{\lambda^k\} \quad \text{with} \quad \lambda^1 \geq \lambda^2 \geq \cdots \geq \lambda^k \geq \cdots \to 0.$$

The main result of [13], [29], [38], [48] is the following:

THEOREM 2.2. – *The sequence of operators S_ϵ converges uniformly to S in the space $\mathcal{L}(L^2(\Omega))$. As a consequence, for a fixed $k \geq 1$,*

$$\lim_{\epsilon \to 0} \lambda_\epsilon^k = \lambda^k$$

and there exists a normalized eigenfunction $v^k \in L^2(\Omega)$ of S, with $\|v^k\|_{L^2(\Omega)} = 1$, associated to each λ^k such that, up to a subsequence,

$$v_\epsilon^k \longrightarrow v^k \text{ strongly in } L^2(\Omega).$$

REMARK 2.3. – *In Theorem 2.2 the convergence of the eigenvectors holds up to a subsequence, even if they are carefully normalized. The reason is that S may have eigenvalues of multiplicity larger than one, implying that a sequence v_ϵ^k may have several accumulation points which are all eigenvectors of S associated to the same eigenvalue.*

REMARK 2.4. – *Theorem 2.2 shows that $\lim_{\epsilon\to 0} \sigma(S_\epsilon) = \sigma(S)$, but it does not say anything on sequences λ_ϵ^k where both ϵ goes to 0 and k goes to $+\infty$ (such sequences go to 0). This latter situation is called a* high frequency limit, *while Theorem 2.2 gives a* low frequency limit. *The goal of the remaining sections of this paper is to describe this high frequency limit.*

Although classical, the proof of Theorem 2.2 contains many useful ideas for the sequel, so we recall it briefly. The uniform convergence of S_ϵ is a straightforward consequence of the following classical result of the homogenization theory, the proof of which may be found in [10], [27], [35].

PROPOSITION 2.5. – *Let f_ϵ be a sequence in $L^2(\Omega)$ which converges weakly to a limit f. Let u_ϵ be the unique solution in $H_0^1(\Omega)$ of*

$$\begin{cases} -\operatorname{div} A\left(x, \frac{x}{\epsilon}\right) \nabla u_\epsilon = f_\epsilon & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The sequence u_ϵ converges weakly in $H_0^1(\Omega)$, and thus strongly in $L^2(\Omega)$ by Rellich theorem, to a limit u which is the unique solution in $H_0^1(\Omega)$ of the homogenized equation

$$\begin{cases} -\operatorname{div} A^*(x) \nabla u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $A^*(x)$ is the homogenized matrix defined by (6).

Proof of Theorem 2.2. – To prove the uniform convergence of S_ϵ to S amounts to check that

$$\|S_\epsilon - S\| = \sup_{\|f\|_{L^2(\Omega)}=1} \|S_\epsilon f - Sf\|_{L^2(\Omega)}$$

goes to zero with ϵ. For fixed ϵ, let f_ϵ be an ϵ-minimizer, *i.e.* a function such that $\|f_\epsilon\|_{L^2(\Omega)} = 1$ and

$$\sup_{\|f\|_{L^2(\Omega)}=1} \|S_\epsilon f - Sf\|_{L^2(\Omega)} \leq \|S_\epsilon f_\epsilon - Sf_\epsilon\|_{L^2(\Omega)} + \epsilon.$$

Since the sequence f_ϵ is bounded in $L^2(\Omega)$, there exists a subsequence, still denoted by ϵ, and a limit f such that the subsequence f_ϵ converges weakly to f in $L^2(\Omega)$. By virtue of Proposition 2.5 the sequence $S_\epsilon f_\epsilon$ converges strongly to Sf in $L^2(\Omega)$. Moreover, since S is a compact operator, Sf_ϵ converges also strongly to Sf in $L^2(\Omega)$. Thus, we have:

$$\sup_{\|f\|_{L^2(\Omega)}=1} \|S_\epsilon f - Sf\|_{L^2(\Omega)} \leq \|S_\epsilon f_\epsilon - Sf\|_{L^2(\Omega)} + \|Sf_\epsilon - Sf\|_{L^2(\Omega)} + \epsilon,$$

which goes to zero with ϵ. This is true for any converging subsequence of f_ϵ. Therefore, this result holds for the entire sequence ϵ.

By the min-max principle, the k^{th} eigenvalue λ_ϵ^k of S_ϵ is defined by

$$\lambda_\epsilon^k = \min_{(f_1,\ldots,f_{k-1})\in L^2(\Omega)} \max_{f\perp\{f_1,\ldots,f_{k-1}\}} \frac{\langle S_\epsilon f, f\rangle_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}},$$

where $\perp$ denotes orthogonality with respect to the usual scalar product in $L^2(\Omega)$. For any $f \in L^2(\Omega)$, we have

$$\frac{\langle S_\epsilon f, f\rangle_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}} - \|S_\epsilon - S\| \leq \frac{\langle Sf, f\rangle_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}} \leq \frac{\langle S_\epsilon f, f\rangle_{L^2(\Omega)}}{\|f\|^2_{L^2(\Omega)}} + \|S_\epsilon - S\|,$$

which implies that

$$\left|\lambda_\epsilon^k - \lambda^k\right| \leq \|S_\epsilon - S\|,$$

thanks to the min-max principle. Thus, the uniform convergence of S_ϵ yields the convergence of each individual eigenvalue, labeled by decreasing order. Now, let v_ϵ^k be a sequence of normalized eigenvectors corresponding to the eigenvalue λ_ϵ^k

$$S_\epsilon v_\epsilon^k = \lambda_\epsilon^k v_\epsilon^k \quad \text{and} \quad \|v_\epsilon^k\|_{L^2(\Omega)} = 1.$$

There exists a subsequence, still denoted by ϵ, and a limit v^k such that the subsequence v_ϵ^k converges weakly to v^k in $L^2(\Omega)$. By virtue of Proposition 2.5 the sequence $S_\epsilon v_\epsilon^k$ converges strongly to Sv^k in $L^2(\Omega)$. Since λ_ϵ^k converges to λ^k, it implies that v_ϵ^k converges strongly to v^k in $L^2(\Omega)$ and that v^k is also a normalized eigenvector corresponding to the eigenvalue λ^k. If the normalizing condition implies the uniqueness of the eigenvector v^k (up to a change of sign), then the entire sequence of eigenvectors v_ϵ^k converges to v^k. But, in case of a multiple eigenvalue λ^k, the convergence of v_ϵ^k to v^k holds merely for a subsequence.

In the course of the proof of Theorem 2.2, we have proved the following Lemma.

LEMMA 2.6. – *Let S_ϵ be a sequence of compact self-adjoint operators acting in $L^2(\Omega)$ and denote their spectrum by $\sigma(S_\epsilon)$. Assume that the sequence S_ϵ converges uniformly to a compact limit operator S with spectrum $\sigma(S)$. Then,*

$$\lim_{\epsilon \to 0} \sigma(S_\epsilon) = \sigma(S). \tag{9}$$

REMARK 2.7. – *The spectral convergence* (20) *has to be understood in the sense of Kuratowsky (or Γ-) convergence for subsets of* ℝ (see e.g. [20]). *Namely, $\sigma(S)$ is the set of all accumulation points λ of sequences $\lambda_\epsilon \in \sigma(S_\epsilon)$ when ϵ goes to zero.*

An interesting question is how can one relax the assumption of uniform convergence of S_ϵ to S and still obtain a result similar to (9) ? In particular, if the sequence S_ϵ converges merely pointwise to S, in the strong or weak topology of $L^2(\Omega)$, what is the limit of the spectrum $\sigma(S_\epsilon)$? Remark first that, in such a case, the limit operator S needs not to be compact. In the case of strong convergence, it turns out that the spectrum of the limit operator is included in the limit spectrum but the inclusion may be strict. In other words, for a strong convergence of operators the spectrum is merely lower semi-continuous. No such result is available for a weak convergence of operators, which is therefore a useless notion concerning spectral convergence.

LEMMA 2.8. – *Let S_ϵ be a sequence of compact self-adjoint operators acting in $L^2(\Omega)$ with spectrum $\sigma(S_\epsilon)$. Assume that the sequence S_ϵ converges strongly to a self-adjoint limit operator S (not necessarily compact) with spectrum $\sigma(S)$ (*i.e. *for each $f \in L^2(\Omega)$, $S_\epsilon f$ converges strongly to Sf in $L^2(\Omega)$). Then,*

$$\lim_{\epsilon \to 0} \sigma(S_\epsilon) \supset \sigma(S).$$

Furthermore, denoting by $(\lambda_\epsilon, f_\epsilon)$ a sequence of eigenvalues and eigenvectors of S_ϵ such that

$$S_\epsilon f_\epsilon = \lambda_\epsilon f_\epsilon, \quad \|f_\epsilon\|_{L^2(\Omega)} = 1, \quad \lim_{\epsilon \to 0} \lambda_\epsilon = \lambda,$$

if λ does not belong to $\sigma(S)$, then the sequence f_ϵ converges weakly to 0 in $L^2(\Omega)$.

Proof. – Let $\lambda \in \sigma(S)$, and assume that λ is not the limit of any sequence of eigenvalues of S_ϵ. This means that there exists a positive constant $\delta > 0$, such that, for sufficiently small ϵ, and for any eigenvalue $\lambda_\epsilon \in \sigma(S_\epsilon)$, one has

$$|\lambda_\epsilon - \lambda| \geq \delta.$$

Obviously, this implies that, for any function $f \in L^2(\Omega)$,

$$\|S_\epsilon f - \lambda f\|_{L^2(\Omega)} \geq \delta \|f\|_{L^2(\Omega)}. \tag{10}$$

Since the convergence of S_ϵ to S is strong, one can pass to the limit in (10) and obtain

$$\|Sf - \lambda f\|_{L^2(\Omega)} \geq \delta \|f\|_{L^2(\Omega)},$$

for any function f, which is a contradiction with the fact that λ belongs to the spectrum of S. Thus, λ is attained as a limit of a sequence $\lambda_\epsilon \in \sigma(S_\epsilon)$.

To complete the proof, it remains to show that, if a sequence of eigenvalues λ_ϵ converges to a limit λ outside $\sigma(S)$, then any associated sequence of eigenvectors f_ϵ converges to zero weakly in $L^2(\Omega)$. The spectral equation is

$$S_\epsilon f_\epsilon = \lambda_\epsilon f_\epsilon. \tag{11}$$

Multiplying (11) by a test function $\phi \in L^2(\Omega)$, and using the symmetry of S_ϵ yields

$$\langle f_\epsilon, S_\epsilon \phi\rangle = \lambda_\epsilon \langle f_\epsilon, \phi\rangle.$$

Thanks to the strong convergence of S_ϵ, we can pass to the limit (up to a subsequence), and denoting by f the weak limit of a subsequence f_ϵ we obtain

$$Sf = \lambda f.$$

Since λ does not belong to $\sigma(S)$, it necessarily implies that the limit f is equal to zero. This is true for any converging subsequence, thus it holds for the entire sequence.

3. Main results

The previous section has investigated the low frequency limit of the spectrum σ_ϵ defined by (3). Theorem 2.2 has given a complete characterization of $\lim\limits_{\epsilon\to 0} \sigma_\epsilon$. However, it says nothing on the high frequency limit which is concerned with sequences of eigenvalues λ_ϵ which go to 0. We now focus on this latter case and try to characterize the renormalized limits $\lim_{\epsilon\to 0} a_\epsilon^{-2}\sigma_\epsilon$ where a_ϵ is a sequence of scales which goes to 0 with ϵ. In other words we are looking to eigenfrequencies $\omega_\epsilon = \lambda_\epsilon^{-1/2}$ which are of the order of a_ϵ^{-1}.

Let us first consider eigenvalues λ_ϵ of the order of ϵ^2, which corresponds to a critical case. To study the renormalized limit $\lim_{\epsilon\to 0}\epsilon^{-2}\sigma_\epsilon$, we introduce a family of limit operators $S_{x,\theta}$ indexed by the macroscopic variable $x \in \overline{\Omega}$ and by the Bloch frequency variable $\theta \in [0,1]^N$. Each operator $S_{x,\theta}$ is defined by:

$$\begin{cases} S_{x,\theta} : L^2_\#(Y) & \longrightarrow & L^2_\#(Y) \\ \qquad \phi & \longrightarrow & u_0, \end{cases} \tag{12}$$

where u_0 is the unique solution in $H^1_\#(Y)$ of

$$\begin{cases} -\operatorname{div}_y\left[A(x,y)\nabla_y\left(u_0(y)e^{2\pi i\theta\cdot y}\right)\right] = \phi(y)e^{2\pi i\theta\cdot y} \text{ in } Y, \\ u_0(y) Y\text{-periodic.} \end{cases} \tag{13}$$

Throughout this paper, the subscript # indicates a space of periodic functions. In the case $\theta = 0$, equation (13) makes sense if $S_{x,0}$ is restricted to the subspace of zero-average functions in $L^2_\#(Y)$. Each $S_{x,\theta}$ is a self-adjoint compact operator in $\mathcal{L}(L^2_\#(Y))$ with spectrum

$$\sigma(S_{x,\theta}) = \{0\} \cup \bigcup_{k\geq 1} \{\lambda^k(x,\theta)\}$$

with $\lambda^1(x,\theta) \geq \lambda^2(x,\theta) \geq \cdots \geq \lambda^k(x,\theta) \geq \cdots \to 0$.

Using the min-max principle and the continuity with respect to x of the matrix $A(x,y)$, it is easy to prove (*see* Proposition 4.12) that each eigenvalue $\lambda^k(x,\theta)$ is a continuous function of $(x,\theta) \in \overline{\Omega} \times Y$. This allows to define the so-called Bloch spectrum σ_{Bloch} as the union of all spectra $\sigma(S_{x,\theta})$

$$\sigma_{\text{Bloch}} = \{0\} \cup \bigcup_{k\geq 1} \left[\min_{(x,\theta)\in\overline{\Omega}\times Y} \lambda^k(x,\theta), \max_{(x,\theta)\in\overline{\Omega}\times Y} \lambda^k(x,\theta) \right].$$

Remark that the Bloch spectrum has a band structure. It could turn out that these bands (*i.e.*, each interval $[\min_{(x,\theta)} \lambda^k(x,\theta), \max_{(x,\theta)} \lambda^k(x,\theta)]$) do overlap. This is the case, for example, when the matrix $A(x,y)$ does not depend on y. However, it is known for some explicit examples that the gaps between bands are not empty (*see* [21]). A similar situation arises in the context of Schrödinger equation (*see e.g.* [41]). The problem of finding conditions on the matrix $A(x,y)$ for the bands to overlap or not is very difficult and not addressed here.

We need also to define a so-called *boundary layer spectrum* σ_{boundary}. Let us consider a sequence of eigenvalues and eigenvectors $(\lambda_\epsilon, v_\epsilon)$ solution of the spectral equation (1). Assume that for a subsequence, still denoted by ϵ, there exists a limit λ such that:

$$\lim_{\epsilon\to 0} \epsilon^{-2}\lambda_\epsilon = \lambda, \quad \|v_\epsilon\|_{L^2(\Omega)} = 1. \tag{14}$$

Then, the limit eigenvalue λ is said to belong to the boundary layer spectrum if, for any positive integer $n \geq 1$, there exists a positive constant $C(n) > 0$ such that

$$\|v_\epsilon d(x,\partial\Omega)^n\|_{L^2(\Omega)} + \epsilon\|(\nabla v_\epsilon) d(x,\partial\Omega)^n\|_{L^2(\Omega)} \leq C(n)\epsilon^n, \tag{15}$$

where $d(x,\partial\Omega)$ is the distance function to the boundary. In other words, the boundary layer spectrum is defined by

$$\sigma_{\text{boundary}} = \left\{\lambda \in \mathbb{R}^+ \mid \exists(\lambda_\epsilon, v_\epsilon) \text{ solutions of (1) satisfying (14), (15)}\right\}. \tag{16}$$

Physically speaking, the boundary layer spectrum corresponds to sequences of eigenvectors concentrating near the boundary. Remark that, compared to the Bloch spectrum, the definition of the boundary layer spectrum is not explicit. It may even depend on the choice of the sequence ϵ (on the contrary of the definition of σ_{Bloch}).

Our main result (announced in [4], [5]) is:

THEOREM 3.1. – *The renormalized limit spectrum is exactly equal to the Bloch and the boundary layer spectra*

$$\lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon = \sigma_{\text{boundary}} \cup \sigma_{\text{Bloch}}. \tag{17}$$

The statement of Theorem (3.1) is somehow weak since it concerns only eigenvalues. However, its proof, which covers sections 4 and 5, gives much more informations. In particular, we exhibit a family of limit operators to which different extensions of the original operator S_ϵ converge strongly in some suitable topology. Then, by Rellich theorem we deduce also a strong convergence of the spectral families which can be interpreted as an "averaged" convergence for the eigenvectors (*see* the original paper [42] or modern textbooks as [28], or [43]). A key feature of Theorem 3.1 is that its proof does not use any labeling of the eigenvalues which is consistent with the obtained densification of the spectrum in the limit as ϵ goes to 0. In a different context (Schrödinger equation in the whole space $\mathbb{R}^N$) related results have been obtained in [25] by a completly different method. In Theorem 3.1, the scaling ϵ^2 of the eigenvalues λ_ϵ can be interpreted as a critical size. Indeed, for any other scaling, we find a simpler result since there is no interaction between the period size ϵ and the frequency size a_ϵ.

THEOREM 3.2. – *Let a_ϵ be a sequence in $\mathbb{R}^+$ which goes to 0 with ϵ and such that, either*

$$\lim_{\epsilon\to 0} \epsilon^{-1} a_\epsilon = 0,$$

or

$$\lim_{\epsilon\to 0} \epsilon^{-1} a_\epsilon = +\infty,$$

then, we have:

$$\lim_{\epsilon\to 0} (a_\epsilon)^{-2} \sigma_\epsilon = \mathbb{R}^+.$$

REMARK 3.3. – *The spectral convergence in Theorems* 3.1 *and* 3.2 *has to be understood in the sense of Kuratowsky (or* Γ*-) convergence for subsets of* $\mathbb{R}$ (see e.g. [20]). *Namely, the limit is the set of all accumulation points λ of renormalized sequences $a_\epsilon^{-2}\lambda_\epsilon$, as ϵ goes to zero, with $\lambda_\epsilon \in \sigma_\epsilon$.*

The proof of Theorem 3.2 is given in section 6. Theorem 3.2 is consistent with Weyl's asymptotic distribution of eigenvalues for the Laplacian. Indeed, if there were no periodic heterogeneities (*i.e.* if the matrix $A(x, y)$ is constant), then Weyl's result would imply that the renormalized limit of the spectrum is always the entire positive half line.

Theorem 3.1 leaves open the question of how characterizing explicitly the boundary layer spectrum. Indeed, our definition of σ_{boundary} is not very enlightening, because it does not characterize this part of the limit of $\epsilon^{-2}\sigma_\epsilon$ as the spectrum of an operator associated with the boundary $\partial\Omega$ of Ω. In particular, it does not say whether σ_{boundary} is empty or included in σ_{Bloch}. There is a subtle point here: the definition of σ_{boundary} depends on the choice of the sequence ϵ. A striking result has recently been obtained by Castro and Zuazua [14] when the sequence ϵ takes all real values close to 0.

THEOREM 3.4. – *Let ϵ be the sequence of all real numbers in the interval $(0, \epsilon_0)$ with $\epsilon_0 > 0$. Then,*

$$\lim_{\epsilon\to 0} \epsilon^{-2} \sigma_\epsilon = \mathbb{R}^+,$$

which means that the boundary layer spectrum σ_{boundary} must necessarily fill the gaps of the Bloch spectrum σ_{Bloch}.

In Theorem 3.4 it is crucial that the sequence ϵ takes all possible values near 0 (*see* its proof in [14]). On the contrary, for a special choice of polygonal domains Ω and discrete (countable) sequences ϵ, we obtain in section 7 a complete characterization of σ_{boundary}

which may not fill any longer the gaps of σ_{Bloch}. However, the general case is still open. Let us assume from now on that Ω is a rectangle with integer dimensions

$$\Omega = \prod_{i=1}^{N}]0; L_i[\quad \text{and} \quad L_i \in \mathbb{N}^* \tag{18}$$

and that the sequence ϵ is exactly

$$\epsilon_n = \frac{1}{n}, \; n \in \mathbb{N}^*. \tag{19}$$

These assumptions imply that, for any ϵ_n, the domain Ω is the union of a finite number of *entire* cells of size ϵ_n. Let Σ be the face of Ω in the plane $x_N = 0$. A generic point x in $\mathbb{R}^N$ is denoted by $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$ and $x_N \in \mathbb{R}$. To define the part of the boundary layer spectrum associated to Σ, we introduce a new periodicity cell which is the semi-infinite band

$$G = Y' \times]0; +\infty[,$$

where $Y' =]0,1[^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. In $L^2_\#(G)$, we define a new family of "boundary layer" limit operators $S_{x',\theta'}$ indexed by the macroscopic variable $x' \in \overline{\Sigma}$ and by the reduced Bloch frequency variable $\theta' \in [0,1]^{N-1}$. Here, $L^2_\#(G)$ denotes the space of squared integrable functions in G which are merely Y'-periodic with respect to y' (and not y_N). Each operator $S_{x',\theta'}$ is defined by:

$$\begin{cases} S_{x',\theta'} : L^2_\#(G) & \longrightarrow L^2_\#(G) \\ \qquad \phi & \longrightarrow u_0, \end{cases}$$

where u_0 is the unique solution of:

$$\begin{cases} -\operatorname{div}_y \left[A((x',0),y) \nabla_y \left(u_0(y) e^{2\pi i \theta' \cdot y'} \right) \right] = \phi(y) e^{2\pi i \theta' \cdot y'} \quad \text{in } G, \\ u_0(y',0) = 0 \\ \lim_{y_N \to +\infty} u_0(y', y_N) = 0 \\ y' \to u_0(y', y_N) \; Y'\text{-periodic.} \end{cases} \tag{20}$$

If $\theta' \neq 0$, $S_{x',\theta'}$ is well defined, as an operator acting in $L^2_\#(G)$, by (20) (remark that the limit behavior of u_0 as y_N goes to infinity has to be understood in the L^2 sense). However, for $\theta' = 0$ it is necessary to shift the spectrum of $S_{x',\theta'}$ by adding a zero-order term in (20) so as to avoid technical difficulties in defining $S_{x',\theta'}$ acting in $L^2_\#(G)$. In any case, $S_{x',\theta'}$ is a self-adjoint *non-compact* operator, and its spectrum $\sigma(S_{x',\theta'})$ is not any longer discrete, but it depends continuously on (x', θ'). Therefore, we can define the boundary layer spectrum associated to the surface Σ

$$\sigma_\Sigma = \bigcup_{x' \in \overline{\Sigma}} \; \bigcup_{\theta' \in [0,1]^{N-1}} \sigma(S_{x',\theta'}),$$

which has again a band structure. Of course, the definition of σ_Σ can be achieved for any face Σ of the rectangle Ω, and a completely similar analysis can be done for all the lower dimensional manifolds (edges, corners, etc.) of which the boundary $\partial\Omega$ is made up. For each type of manifold, a different family of limit problems arise which are straightforward generalizations of (20). For example, in two space dimensions, the corners of Ω give rise to a limit problem in the quarter of space $\mathbb{R}^+ \times \mathbb{R}^+$ (*see* subsection 7.3). Finally, our last main result is:

THEOREM 3.5. – *Under assumptions* (18) *and* (19), *the renormalized limit of the sequence of spectra* $\epsilon_n^{-2}\sigma_{\epsilon_n}$ *is precisely given by*

$$\lim_{\epsilon_n\to 0} \epsilon_n^{-2}\sigma_{\epsilon_n} = \sigma_{\text{Bloch}} \cup \sigma_{\partial\Omega},$$

with the notation

$$\sigma_{\partial\Omega} = \bigcup_{\Sigma\subset\partial\Omega} \sigma_\Sigma$$

where the union is over all hypersurfaces and lower dimensional manifolds composing the boundary $\partial\Omega$.

Theorems 3.1 and 3.5 are proved in section 7. The difference between Theorem 3.1 and Theorem 3.5 is that the latter boundary layer spectrum $\sigma_{\partial\Omega}$ is explicitly defined for the specific sequence of parameters ϵ_n as the spectrum of a family of limit operators, while the first of these boundary layer spectra, σ_{boundary} was indirectly defined for any sequence ϵ but not explicitly characterized. Remark also that we do not prove that $\sigma_{\partial\Omega}$ and σ_{boundary} coincide, but merely that $\sigma_{\text{boundary}} \subset \sigma_{\partial\Omega}$. We believe that the inclusion is usually strict, even if a more precise definition of σ_{boundary} is used.

4. Bloch wave homogenization

This section is devoted to the first part of the proof of Theorem 3.1. By means of a new method of homogenization, called homogenization by Bloch waves, we shall prove that

$$\sigma_{\text{Bloch}} \subset \lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon.$$

This convergence result holds for any choice of the sequence ϵ. To analyze the behavior of eigenvalues of the order of ϵ^2, the spectral problem (1) is rewritten as follows: find $(\mu_\epsilon, v_\epsilon) \in \mathbb{R}^+ \times H_0^1(\Omega), v_\epsilon \not\equiv 0$, such that:

$$\left\{ \begin{array}{ll} -\epsilon^2 \operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla v_\epsilon\right] + v_\epsilon = \frac{1}{\mu_\epsilon} v_\epsilon & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{21}$$

Passing from (1) to (21) leaves invariant the eigenfunctions and change the eigenvalues λ_ϵ^k into μ_ϵ^k (labeled in decreasing order) defined by

$$\mu_\epsilon^k = \lambda_\epsilon^k/(\epsilon^2 + \lambda_\epsilon^k). \tag{22}$$

This has the effect that $\mu_\epsilon^k \sim 1$ if $\lambda_\epsilon^k \sim \epsilon^2$.

To (21) is associated a new operator $\tilde{S}_\epsilon \in \mathcal{L}(L^2(\Omega))$ defined by:

$$\left\{\begin{array}{lcl}\tilde{S}_\epsilon : L^2(\Omega) & \longrightarrow & L^2(\Omega)\\ f & \longrightarrow & u_\epsilon,\end{array}\right. \tag{23}$$

where u_ϵ is the unique solution in $H_0^1(\Omega)$ of:

$$\left\{\begin{array}{ll}-\epsilon^2 \operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla u_\epsilon\right] + u_\epsilon = f & \text{in } \Omega,\\ u_\epsilon = 0 & \text{on } \partial\Omega.\end{array}\right. \tag{24}$$

Of course, for fixed ϵ, $\tilde{S}_\epsilon$ is still a self-adjoint compact operator. We denote by $\sigma(\tilde{S}_\epsilon)$ its spectrum made of a countable sequence of eigenvalues converging to 0, plus the limit point 0,

$$\sigma(\tilde{S}_\epsilon) = \{0\} \cup \bigcup_{k\geq 1}\{\mu_\epsilon^k\} \quad \text{with} \quad \mu_\epsilon^1 \geq \mu_\epsilon^2 \geq \cdots \geq \mu_\epsilon^k \geq \cdots \to 0.$$

REMARK 4.1. – *It is an easy exercise in homogenization to show that the solution u_ϵ of (24) has a tendency to periodically oscillate like $u_0\left(x, \frac{x}{\epsilon}\right)$ when ϵ is small, and therefore it converges merely weakly in $L^2(\Omega)$. Furthermore, its weak limit is easily shown to be nothing else than f. This implies that $\tilde{S}_\epsilon$ converges weakly to the identity operator in $\mathcal{L}(L^2(\Omega))$. In terms of spectral convergence, we cannot deduce anything from this weak convergence. In any case, the limit operator (the identity) does not contain much information left from the sequence $\tilde{S}_\epsilon$.*

Since the solution u_ϵ of (24) behaves like an oscillating function $u_0\left(x, \frac{x}{\epsilon}\right)$, the key idea in order to obtain a strong convergence of the sequence $\tilde{S}_\epsilon$ is to extend it to a larger space of "two-scale oscillating" functions, capable of describing this oscillating behavior. In other words, we first embed $L^2(\Omega)$ in the larger space $L^2(\Omega \times Y)$ of functions $\phi(x, y)$ of two variables $x \in \Omega$ (the slow variable) and $y \in Y = [0,1]^N$ (the fast periodic variable). For reasons that will be clear afterwards (mainly because of the Bloch waves decomposition), we actually extend the operator $\tilde{S}_\epsilon$ to the space $L^2(\Omega; L^2_\#(KY))$ where $K \geq 1$ is a given positive integer, and KY denotes the cube $[0, K]^N$. In other words, we use two-scale oscillating functions on a larger period KY. More precisely, we define an extended operator $S_\epsilon^K \in \mathcal{L}(L^2(\Omega; L^2_\#(KY)))$ by

$$S_\epsilon^K = E_\epsilon^K \tilde{S}_\epsilon P_\epsilon^K,$$

where P_ϵ^K and E_ϵ^K are respectively a projection from $L^2(\Omega; L^2_\#(KY))$ onto $L^2(\Omega)$ and an extension from $L^2(\Omega)$ into $L^2(\Omega; L^2_\#(KY))$. To be sure that S_ϵ^K is still self-adjoint, we ask P_ϵ^K and E_ϵ^K to be adjoint one from the other. To insure that $\tilde{S}_\epsilon$ and S_ϵ^K have the same spectrum, we ask the product $P_\epsilon^K E_\epsilon^K$ to be equal to the identity in $L^2(\Omega)$. The Hilbert space $L^2(\Omega; L^2_\#(KY))$ is equipped with the scalar product

$$\langle \phi, \psi \rangle = K^{-N} \int_\Omega \int_{KY} \phi(x, y)\psi(x, y)dxdy.$$

To build such extension and projection operators, we introduce a regular mesh of size $K\epsilon$ on the domain Ω: let $(Y_i^\epsilon)_{1\leq i\leq n(\epsilon)}$ be a family of non-overlapping cells of the type $[0;K\epsilon]^N$ covering Ω (the number of cells is $n(\epsilon)$ which is of the order of $(K\epsilon)^{-N}|\Omega|$). We denote by x_i^ϵ the origin of each cell Y_i^ϵ and by $\chi_i^\epsilon(x)$ its characteristic function. Defining a projection operator by

$$\text{(25)}\qquad \begin{cases} P_\epsilon^K : L^2(\Omega; L^2_\#(KY)) & \longrightarrow L^2(\Omega) \\ \phi(x,y) & \longmapsto \sum_{i=1}^{n(\epsilon)} \chi_i^\epsilon(x) \frac{1}{(K\epsilon)^N} \int_{Y_i^\epsilon} \phi(x', \frac{x}{\epsilon})dx', \end{cases}$$

and an extension operator by

$$\text{(26)}\qquad \begin{cases} E_\epsilon^K : L^2(\Omega) & \longrightarrow L^2(\Omega; L^2_\#(KY)) \\ f(x) & \longmapsto \sum_{i=1}^{n(\epsilon)} \chi_i^\epsilon(x) f(x_i^\epsilon + \epsilon y), \end{cases}$$

their announced properties are checked in the following:

LEMMA 4.2. – *The operators P_ϵ^K and E_ϵ^K defined by* (25) *and* (26) *satisfy*

$$P_\epsilon^K E_\epsilon^K = \mathrm{Id}_{L^2(\Omega)} \quad \text{and} \quad \left(P_\epsilon^K\right)^* = E_\epsilon^K.$$

Furthermore, the product $E_\epsilon^K P_\epsilon^K$ converges strongly to the identity in the space $\mathcal{L}\big(L^2(\Omega; L^2_\#(KY))\big)$.

REMARK 4.3. – *At first look, the most natural projection operator from the space $L^2(\Omega; L^2_\#(KY))$ onto $L^2(\Omega)$ seems to be the application that maps $\phi\left(x, \frac{x}{\epsilon}\right)$ to any function $\phi(x,y)$. Unfortunately, this is not a continuous operator. There is even no guaranty that $\phi\left(x, \frac{x}{\epsilon}\right)$ is measurable on Ω for a general ϕ. This explains the complicated definition of the projection operator P_ϵ^K which is as close as possible to this idea, while having good functional properties.*

Proof. – A simple computation yields that $P_\epsilon^K E_\epsilon^K = \mathrm{Id}_{L^2(\Omega)}$. Furthermore,

$$\int_\Omega \int_{KY} \phi(x,y)(E_\epsilon^K f)(x,y)dxdy = \sum_{i=1}^{n(\epsilon)} \int_{Y_i^\epsilon} \int_{KY} \phi(x,y) f(x_i^\epsilon + \epsilon y)dxdy$$
$$= \sum_{i=1}^{n(\epsilon)} \int_{KY} f(x_i^\epsilon + \epsilon y) \int_{Y_i^\epsilon} \phi(x', y)dx'dy = \frac{1}{\epsilon^N} \sum_{i=1}^{n(\epsilon)} \int_{Y_i^\epsilon} f(x) \int_{Y_i^\epsilon} \phi(x', \frac{x}{\epsilon})dx'dx$$
$$= K^N \int_\Omega f(x)\Big(\sum_{i=1}^{n(\epsilon)} \chi_i^\epsilon(x) \frac{1}{(K\epsilon)^N} \int_{Y_i^\epsilon} \phi(x', \frac{x}{\epsilon})dx'\Big)dx = K^N \int_\Omega f(x) P_\epsilon^K \phi(x)dx,$$

which proves $(P_\epsilon^K)^* = E_\epsilon^K$. A similar computation shows that

$$\big(E_\epsilon^K P_\epsilon^K \phi\big)(x,y) = \sum_{i=1}^{n(\epsilon)} \chi_i^\epsilon(x) \frac{1}{(K\epsilon)^N} \int_{Y_i^\epsilon} \phi(x', y)dx',$$

i.e., $E_\epsilon^K P_\epsilon^K$ is the projection operator in $L^2(\Omega; L^2_\#(KY))$ on piecewise constant functions in x in each cell Y_i^ϵ. As is well-known, such a projection $E_\epsilon^K P_\epsilon^K$ converges strongly to the identity.

PROPOSITION 4.4. – *The sequence S_ϵ^K converges strongly to a self-adjoint limit operator S^K in the sense that, for any $\phi(x,y) \in L^2(\Omega; L^2_\#(KY))$, $S_\epsilon^K \phi$ converges strongly to $S^K\phi$ in $L^2(\Omega; L^2_\#(KY))$ and $S^K\phi = u^K$ is the unique solution in the space $L^2(\Omega; H^1_\#(KY))$ of*

$$\begin{cases} -\operatorname{div}_y\left[A(x,y)\nabla_y u^K\right] + u^K = \phi & \text{in} \quad \Omega \times KY \\ y \to u^K(x,y) \ KY\text{-periodic.} \end{cases} \tag{27}$$

We shall prove below (*see* Proposition 4.12) that S^K is a non-compact operator in $L^2\big(\Omega; L^2_\#(KY)\big)$. Therefore, the convergence of S_ϵ^K to S^K cannot be uniform since S_ϵ^K is compact, but not S^K. Thus, from Proposition 4.4, we can only deduce the lower semi-continuity of the spectrum (*see* Lemma 2.8).

COROLLARY 4.5. – *The spectrum $\sigma(S^K)$ of S^K satisfies*

$$\sigma(S^K) \subset \lim_{\epsilon\to 0} \sigma(\tilde{S}_\epsilon).$$

Furthermore, as a consequence of Rellich theorem (see e.g. [42], [28], [43]), *for any μ which is not an eigenvalue of S^K, the spectral family $\mathcal{E}_\epsilon^K(\mu)$ of S_ϵ^K converges strongly to that $\mathcal{E}^K(\mu)$ of S^K in $L^2\big(\Omega; L^2_\#(KY)\big)$.*

The key ingredient in the proof of Proposition 4.4 is the notion of two-scale convergence introduced in [2], [36], that we briefly recall in the following

PROPOSITION 4.6.

(1) *Let u_ϵ be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence, still denoted by ϵ, and a limit $u_0(x,y) \in L^2(\Omega; L^2_\#(KY))$ such that u_ϵ* two-scale converges *weakly to u_0 in the sense that*

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(x)\phi\Big(x, \frac{x}{\epsilon}\Big)dx = \frac{1}{|KY|}\int_\Omega\int_{KY} u_0(x,y)\phi(x,y)\,dx\,dy$$

for all functions $\phi(x,y) \in L^2(\Omega; C_\#(KY))$.

(2) *Let u_ϵ be a sequence of functions in $L^2(\Omega)$ which two-scale converges weakly to a limit $u_0(x,y) \in L^2(\Omega; L^2_\#(KY))$. Assume further that*

$$\lim_{\epsilon\to 0} \|u_\epsilon\|^2_{L^2(\Omega)} = \frac{1}{|KY|}\|u_0\|^2_{L^2(\Omega; L^2_\#(KY))}.$$

Then u_ϵ is said to two-scale converge strongly *to its limit u_0 in the sense that, for any sequence v_ϵ which two-scale converges weakly to a limit $v_0(x,y) \in L^2(\Omega; L^2_\#(KY))$, we have*

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon(x)v_\epsilon(x)\phi\Big(x, \frac{x}{\epsilon}\Big)dx = \frac{1}{|KY|}\int_\Omega\int_{KY} u_0(x,y)v_0(x,y)\phi(x,y)\,dx\,dy,$$

for all smooth functions $\phi(x,y) \in C\big(\overline{\Omega}; C_\#(KY)\big)$.

(3) *Let u_ϵ be a bounded sequence in $L^2(\Omega)$ such that $\epsilon\nabla u_\epsilon$ is also bounded in $L^2(\Omega)^N$. Then there exists a subsequence, still denoted by ϵ, and a limit $u_0(x,y) \in L^2(\Omega; H^1_\#(KY))$ such that u_ϵ two-scale converges to $u_0(x,y)$ and $\epsilon\nabla u_\epsilon$ two-scale converges to $\nabla_y u_0(x,y)$.*

Another technical Lemma is required before the proof of Proposition 4.4.

LEMMA 4.7. –

(1) *Let $\phi(x,y)$ be a function in $L^2(\Omega; L^2_\#(KY))$. Then the sequence $(P^K_\epsilon \phi)(x)$ two-scale converges strongly to $\phi(x,y)$.*

(2) *Let $\phi_\epsilon(x,y)$ be a sequence converging weakly to $\phi(x,y)$ in $L^2(\Omega; L^2_\#(KY))$. Then the sequence $(P^K_\epsilon \phi_\epsilon)(x)$ two-scale converges weakly to $\phi(x,y)$.*

Proof. – To prove 2, let $\theta(x,y)$ be a smooth, KY-periodic function. We have:

$$\int_\Omega (P^K_\epsilon \phi^\epsilon)(x)\theta(x,\frac{x}{\epsilon})dx = \frac{1}{K^N}\int_\Omega\int_{KY} \phi^\epsilon(x,y)E^K_\epsilon[\theta(x,\frac{x}{\epsilon})]dxdy.$$

Furthermore,

$$E^K_\epsilon[\theta(x,\frac{x}{\epsilon})] = \theta(x^\epsilon_i + \epsilon y, y)$$

in each cell Y^ϵ_i of the type $[0;K\epsilon]^N$. Here, x^ϵ_i is the origin of Y^ϵ_i. Since $\theta(x,y)$ is a smooth function, it is easily seen that $E^K_\epsilon[\theta(x,\frac{x}{\epsilon})]$ converges strongly to $\theta(x,y)$ in $L^2(\Omega; L^2_\#(KY))$, which completes the proof of 2.

To obtain 1, it remains to prove that, for a fixed test function ϕ, $||P^K_\epsilon\phi||_{L^2(\Omega)}$ converges to $\frac{1}{K^{N/2}}||\phi||_{L^2(\Omega\times KY)}$. Thanks to Lemma 4.2, we have

$$||P^K_\epsilon\phi||^2_{L^2(\Omega)} = \frac{1}{K^N}\int_\Omega\int_{KY}(E^K_\epsilon P^K_\epsilon\phi)\phi dxdy$$

and $E^K_\epsilon P^K_\epsilon$ converges strongly to the identity. This proves the desired result.

Proof of Proposition 4.4. – Let $\psi_\epsilon(x,y)$ be a sequence converging weakly to $\psi(x,y)$ in $L^2(\Omega; L^2_\#(KY))$. For any $\phi \in L^2(\Omega; L^2_\#(KY))$, we need to show that

$$\lim_{\epsilon\to 0}\int_\Omega\int_{KY}(S^K_\epsilon\phi)\psi_\epsilon dxdy = \int_\Omega\int_{KY}(S^K\phi)\psi dxdy.$$

By definition of S^K_ϵ, one has

$$\frac{1}{K^N}\int_\Omega\int_{KY}(S^K_\epsilon\phi)\psi_\epsilon dxdy = \int_\Omega(\tilde{S}_\epsilon P^K_\epsilon\phi)\big((E^K_\epsilon)^*\psi_\epsilon\big)dx = \int_\Omega u_\epsilon(P^K_\epsilon\psi_\epsilon)dx, \tag{28}$$

where u_ϵ is now the solution of

$$\begin{cases} -\epsilon^2\,\mathrm{div}A(x,\frac{x}{\epsilon})\nabla u_\epsilon + u_\epsilon = P^K_\epsilon[\phi(x,y)] & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{29}$$

By Lemma 4.8 below, u_ϵ two-scale converges strongly to the solution u^K of (27). By Lemma 4.7, the sequence $P_\epsilon^K \psi_\epsilon$ two-scale converges weakly to ψ. Then, by Proposition 4.6, we can pass to the limit in (28)

$$\lim_{\epsilon\to 0} \int_\Omega u_\epsilon (P_\epsilon^K \psi_\epsilon) dx = \frac{1}{K^N} \int_\Omega \int_{KY} u^K \psi dx dy = \frac{1}{K^N} \int_\Omega \int_{KY} (S^K \phi)\psi dx dy,$$

which concludes the proof since the map $\phi \to u^K$ is obviously continuous self-adjoint in $L^2(\Omega; L^2_\#(KY))$.

LEMMA 4.8. – *The solution* u_ϵ *of* (29) *two-scale converges to* $u^K(x,y)$ *which is the unique solution in* $L^2(\Omega; H^1_\#(KY))$ *of*

$$\begin{cases} -\operatorname{div}_y \left[A(x,y) \nabla_y u^K \right] + u^K = \phi \textit{in} \quad \Omega \times KY \\ y \to u^K(x,y)\ KY\textit{-periodic.} \end{cases}$$

Furthermore, u_ϵ *two-scale converges strongly to* $u^K(x,y)$, i.e.:

$$\lim_{\epsilon\to 0} \|u_\epsilon\|_{L^2(\Omega)} = \frac{1}{K^{N/2}} \|u^K\|_{L^2(\Omega\times KY)}.$$

Proof. – The following *a priori* estimate is easily derived from equation (29):

$$\|u_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla u_\epsilon\|_{L^2(\Omega)^N} \le C.$$

Then, there exists a limit $u^K(x,y) \in L^2(\Omega; H^1_\#(KY))$, such that, up to a subsequence, u_ϵ and $\epsilon \nabla u_\epsilon$ two-scale converge respectively to $u^K(x,y)$ and $\nabla_y u^K(x,y)$. Multiplying equation (29) by a test function $\theta(x, \frac{x}{\epsilon})$, where $\theta(x,y)$ is a smooth, KY-periodic function, we pass to the limit and get:

$$\int_\Omega \int_{KY} A(x,y) \nabla_y u^K \nabla_y \theta dx dy + \int_\Omega \int_{KY} u^K \theta dx dy = \int_\Omega \int_{KY} \phi \theta dx dy,$$

which is nothing else than the variational formulation of the limit problem which clearly admits a unique solution. The limit u^K is therefore unique, and the entire subsequence u_ϵ two-scale converges to u^K.

Besides, multiplying the equation (29) by u_ϵ, we obtain

$$\int_\Omega \epsilon^2 A(x, \frac{x}{\epsilon}) \nabla u_\epsilon \cdot \nabla u_\epsilon + \int_\Omega |u_\epsilon|^2 = \int_\Omega P_\epsilon^K \phi u_\epsilon$$

which, by virtue of Lemma 4.7, converges to

$$\frac{1}{K^N} \int_\Omega \int_{KY} \phi u^K dx dy = \frac{1}{K^N} \int_\Omega \int_{KY} \left[A(x,y) \nabla_y u^K \cdot \nabla_y u^K + |u^K|^2 \right] dx dy.$$

Then, using the lower semi-continuity of the two-scale convergence (*see* [2]), we conclude that

$$\lim_{\epsilon\to 0} \int_\Omega |u_\epsilon|^2 = \frac{1}{K^N} \int_\Omega \int_{KY} |u^K|^2 dx dy,$$

which finishes the proof of Lemma 4.8.

To compute the spectrum of $\sigma(S^K)$, we use a discrete Bloch waves decomposition in $L^2_\#(KY)$ (*see* [1], or [10], [12], [22], [23], [37], [41], [49] in the continuous case). This Bloch decomposition allows to diagonalize S^K.

LEMMA 4.9. – *For any function $\phi(y) \in L^2_\#(KY)$ there exists a unique family $\{\phi_j(y)\} \in L^2_\#(Y)^{K^N}$, indexed by a multi-index j whose N components belong to $\{0,\dots,K-1\}$, such that*

$$\phi(y) = \sum_{0\le j\le K-1} \phi_j(y) e^{2\pi i \frac{j\cdot y}{K}} \tag{30}$$

and

$$\frac{1}{K^N}\int_{KY} |\phi|^2 dy = \sum_{0\le j\le K-1} \int_Y |\phi_j|^2 dy.$$

Furthermore, if $\psi(y)$ is another function in $L^2_\#(KY)$ with Bloch components $\{\psi_j(y)\} \in L^2_\#(Y)^{K^N}$, we have

$$\frac{1}{K^N}\int_{KY} \phi\overline{\psi} dy = \sum_{0\le j\le K-1} \int_Y \phi_j \overline{\psi}_j dy.$$

This decomposition, denoted by $\mathcal{B}$, defines a unitary isometry from $L^2_\#(KY)$ into $L^2_\#(Y)^{K^N}$.

REMARK 4.10. – *Even if the function $\phi \in L^2_\#(KY)$ is real-valued, its Bloch components $\phi_j \in L^2_\#(Y)$ are complex-valued. Therefore, from now on all functions are supposed to be complex-valued. To simplify the exposition, for any functional space we shall use the same notation for its real or complex-valued version.*

Proof. – For each multi-index $j = 0, ..., K-1$, let us define ϕ_j in $L^2_\#(Y)$ by

$$\phi_j(y) = \frac{1}{K^N}\sum_{j'=0}^{K-1} \phi(y+j') e^{-2\pi i \frac{(y+j')\cdot j}{K}}.$$

It suffices now to check that (30) holds true with the above definition of ϕ_j:

$$\frac{1}{K^N}\sum_{j=0}^{K-1}\sum_{j'=0}^{K-1} \phi(y+j') e^{-2\pi i \frac{j'\cdot j}{K}} = \sum_{j'=0}^{K-1}\phi(y+j')\left(\frac{1}{K^N}\sum_{j=0}^{K-1} e^{-2\pi i\frac{j'\cdot j}{K}}\right). \tag{31}$$

If $j' = 0$, the expression between brackets in the right hand side of (31) is equal to 1. If $j' \neq 0$, it is equal to 0, thanks to a well-known property of the K-th roots of 1 in the complex plane. This proves (30). Parseval and Plancherel formulae are obtained in a similar fashion.

From Lemma 4.9, we easily deduce the following:

PROPOSITION 4.11. – *The operator S^K can be diagonalized as*

$$S^K = \mathcal{B}^* T^K \mathcal{B} \quad \textit{with} \quad T^K = \mathrm{diag}\left[(T_{j/K})_{0\le j\le K-1}\right]$$

where, for each Bloch frequency $\theta = j/K$, T_θ is a self-adjoint non-compact operator defined in $\mathcal{L}\big(L^2(\Omega; L^2_\#(Y))\big)$ by

$$\left\{ \begin{array}{lcl} T_\theta : L^2(\Omega; L^2_\#(Y)) & \longrightarrow & L^2(\Omega; L^2_\#(Y)) \\ \phi & \longrightarrow & u \end{array} \right. \tag{32}$$

where $u(x,y)$ is the unique solution in $L^2(\Omega; H^1_\#(Y))$ of

$$-\operatorname{div}_y\big[A(x,y)\nabla_y\big(ue^{2\pi i\theta\cdot y}\big)\big] + ue^{2\pi i\theta\cdot y} = \phi e^{2\pi i\theta\cdot y} \quad in \quad \Omega\times Y.$$

Consequently the spectrum of S^K is

$$\sigma(S^K) = \bigcup_{0\le j\le K-1} \sigma(T_{\frac{j}{K}}).$$

Proof. – To diagonalize S^K, we apply the Bloch wave decomposition to the variational formulation of equation (27) defining S^K. For any $\phi(x,y)$ in $L^2(\Omega\times KY)$, $S^K\phi$ is defined as the solution $u(x,y)$ of

$$\int_\Omega\int_{KY} A(x,y)\nabla_y u(x,y)\nabla_y\overline{\psi}(x,y) + \int_\Omega\int_{KY} u(x,y)\overline{\psi}(x,y) = \int_\Omega\int_{KY} \phi(x,y)\overline{\psi}(x,y),$$

where ψ is a test function in $L^2(\Omega; H^1_\#(KY))$. Applying the Bloch decomposition operator $\mathcal{B}$ to both u and ψ, we get

$$\mathcal{B}u = \sum_{0\le j\le K-1} u_{\frac{j}{K}}(x,y)e^{2i\pi\frac{j\cdot y}{K}}, \quad \mathcal{B}\psi = \sum_{0\le j\le K-1} \psi_{\frac{j}{K}}(x,y)e^{2i\pi\frac{j\cdot y}{K}}$$

and, since $A(x,y)$ is Y-periodic,

$$\begin{aligned}\sum_{0\le j\le K-1}\int_\Omega\int_Y \Big(A(x,y)\nabla_y\Big(u_{\frac{j}{K}}e^{2\pi i\frac{j}{K}\cdot y}\Big)\cdot\nabla_y\Big(\overline{\psi}_{\frac{j}{K}}e^{-2\pi i\frac{j}{K}\cdot y}\Big) + u_{\frac{j}{K}}\overline{\psi}_{\frac{j}{K}}\Big) \\ = \sum_{0\le j\le K-1}\int_\Omega\int_Y \phi_{\frac{j}{K}}\overline{\psi}_{\frac{j}{K}}.\end{aligned} \tag{33}$$

For each Bloch frequency $\frac{j}{K}$, (33) is nothing but the variational formulation of the equation defining $T_{\frac{j}{K}}\phi_{\frac{j}{K}}$. Therefore, $\Big(T_{\frac{j}{K}}\Big)_{0\le j\le K-1} = \mathcal{B}S^K\mathcal{B}^*$.

PROPOSITION 4.12. – *For any fixed $x\in\overline{\Omega}$ and $\theta\in Y$ we introduce an operator $T_{\theta,x}$ acting on $L^2_\#(Y)$, defined by*

$$\left\{ \begin{array}{lcl} T_{\theta,x} : L^2_\#(Y) & \longrightarrow & L^2_\#(Y) \\ \phi & \longrightarrow & u, \end{array} \right. \tag{34}$$

where $u(y)$ is the unique solution in $H^1_\#(Y)$ of

$$-\operatorname{div}_y\big[A(x,y)\nabla_y\big(ue^{2\pi i\theta\cdot y}\big)\big] + ue^{2\pi i\theta\cdot y} = \phi e^{2\pi i\theta\cdot y} \quad in \quad Y. \tag{35}$$

Then $T_{\theta,x}$ is a self-adjoint compact operator and its spectrum is

$$\sigma(T_{\theta,x}) = \{0\} \bigcup_{k\geq 1} \{\mu^k(\theta,x)\},$$

where each eigenvalue $\mu^k(\theta,x)$ is continuous with respect to $(\theta,x)\in Y\times\overline{\Omega}$. Finally, the operator T_θ defined by (32) *is non-compact and its spectrum is*

$$\sigma(T_\theta) = \bigcup_{x\in\overline{\Omega}} \sigma(T_{\theta,x}) = \{0\} \bigcup_{k\geq 1} [\min_{x\in\overline{\Omega}} \mu^k(\theta,x), \max_{x\in\overline{\Omega}} \mu^k(\theta,x)].$$

REMARK 4.13. – *We recognize in the operators $T_{\theta,x}$ a simple transformation of the operators $S_{x,\theta}$ defined by* (12) *since, due to the change of variables* (22)*, their eigenvalues are related by*

$$\mu^k = \frac{\lambda^k}{(1+\lambda^k)}. \tag{36}$$

Proof. – Clearly, each operator $T_{\theta,x}$ is self-adjoint compact. Therefore, its spectrum is discrete, and labeling the eigenvalues in decreasing order it is given by

$$\tilde{\sigma}_{\theta,x} = \{0\} \cup \bigcup_{k\geq 1} \{\mu^k(\theta,x)\}$$

with

$$\mu^1(\theta,x) \geq \mu^2(\theta,x) \geq \cdots \geq \mu^k(\theta,x) \geq \cdots \to 0.$$

Multiplying equation (35) by $e^{-2\pi i\theta\cdot y}$, yields a new definition of $T_{\theta,x}$ which has the effect that both x and θ appear as parameters in the coefficient matrix. More precisely, $u(y) = T_{\theta,x}\phi(y)$ is the unique solution in $H^1_\#(Y)$ of

$$\begin{cases} -(\nabla_y - 2i\pi\theta)A(x,y)(\nabla_y + 2i\pi\theta)u(y) + u(y) = \phi(y) \text{in } Y \\ y \to u(y) \ Y\text{-periodic.} \end{cases}$$

The eigenvalues are then characterized by the min-max formula:

$$\mu^k(\theta,x) = \min_{\substack{F\subset L^2_\#(Y)\\ \dim F = k}} \max_{u\in F\cap H^1_\#(Y)} \frac{\int_Y A(x,y)(\nabla_y u + 2i\pi\theta u)\cdot\overline{(\nabla_y u + 2i\pi\theta u)} + \int_Y |u|^2}{\int_Y |u|^2}$$

This implies that $\mu^k(x,\theta)$ is continuous (and even Lipschitz) as the min-max of continuous functions as remarked by P. Gérard [24]. Here, we have used the assumption that $x \to A(x,y)$ is continuous in $\overline{\Omega}$. To prove that T_θ is non-compact and compute its spectrum

$$\sigma(T_\theta) = \bigcup_{x\in\overline{\Omega}} \sigma(T_{\theta,x}),$$

we use the Weyl criterion. For any eigenvalue μ in $\sigma(T_{\theta,x_0})$, with eigenvector $u(y)$, a sequence u_n of almost eigenvectors for T_θ is defined by $u_n(x,y) = \phi_n(x)u(y)$, where

$\phi_n(x)$ is a smooth function concentrating at the point x_0 with $\|\phi_n\|_{L^2(\Omega)} = 1$. It is not difficult to check Weyl's criterion which implies that μ belongs to the essential spectrum of T_θ.

Conversely, if $\mu \notin \sigma(T_{\theta,x})$ for any $x \in \overline{\Omega}$, we have

$$\|(T_{\theta,x} - \mu Id)^{-1}\|_{\mathcal{L}(L^2(Y))} \leq C < +\infty,$$

which implies, since $\bigcup_{x\in\overline{\Omega}} \sigma(T_{\theta,x})$ is a closed set, that

$$\|(T_\theta - \mu Id)^{-1}\|_{\mathcal{L}(L^2(\Omega\times Y))} < +\infty$$

and hence, μ does not belong to the spectrum of T_θ.

THEOREM 4.14. – *When K goes to $+\infty$, we have:*

$$\lim_{K\to+\infty} \sigma(S^K) = \bigcup_{x\in\overline{\Omega},\ \theta\in Y} \sigma(T_{\theta,x}) = \{0\} \bigcup_{k\geq 1} [\min_{x\in\overline{\Omega},\ \theta\in Y} \mu^k(\theta,x), \max_{x\in\overline{\Omega},\ \theta\in Y} \mu^k(\theta,x)].$$

Since σ_{Bloch} and $\lim_{K\to+\infty} \sigma(S^K)$ are related through the change of variables (36), *we deduce that, for any sequence ϵ converging to 0, $\sigma_{\text{Bloch}} \subset \lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon$.*

Proof. – Recall that the choice of the integer K is arbitrary, and that we proved

$$\lim_{\epsilon\to 0} \sigma(\tilde{S}_\epsilon) \supset \lim_{K\to\infty} \sigma(S^K) = \bigcup_{0\leq j\leq K-1} \sigma(T_{\frac{j}{K}}).$$

Since the spectrum $\sigma(T_\theta)$ is continuous with respect to θ, letting K go to $+\infty$ yields the desired result.

REMARK 4.15. – *Let us indicate that this method of Bloch wave homogenization has already been applied to a different model of fluid-solid structure* (see [6]).

5. Completeness

This section is devoted to the proof of the second part of Theorem 3.1. In the previous section we proved that

$$\sigma_{\text{Bloch}} \subset \lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon.$$

Here, we prove that the difference between the limit renormalized spectrum and the Bloch spectrum is the so-called boundary layer spectrum

$$\lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon = \sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}}.$$

A precise definition of the boundary layer spectrum σ_{boundary} is given below in Definition 5.1.

To characterize the limit of the renormalized spectrum $\epsilon^{-2}\sigma_\epsilon$, we consider a sequence of eigenvalues μ_ϵ and eigenvectors $v_\epsilon \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\|v_\epsilon\|_{L^2(\Omega)} = 1, \quad \lim_{\epsilon\to 0} \mu_\epsilon = \mu \quad \text{and} \tag{37}$$

$$\begin{cases} -\epsilon^2 \operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla v_\epsilon\right] + v_\epsilon = \frac{1}{\mu_\epsilon} v_\epsilon & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$

We introduce the distance function to the boundary, denoted by $d(x, \partial\Omega)$, and defined by

$$d(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y| \quad \forall x \in \Omega.$$

Assuming that the boundary $\partial\Omega$ is Lipschitz, the distance function $d(x, \partial\Omega)$ belongs to $W_0^{1,\infty}(\Omega)$. We are now in a position to define the boundary layer spectrum.

DEFINITION 5.1. – *The boundary layer spectrum* σ_{boundary} *is defined as the set of all limit eigenvalues* μ *such that any corresponding sequence of eigenvectors* v_ϵ *satisfying* (37) *and* (38) *has also the property that, for any positive integer* $n \geq 1$, *there exists a positive constant* $C(n) > 0$ *and the entire sequence satisfies*

$$\|v_\epsilon d(x, \partial\Omega)^n\|_{L^2(\Omega)} + \epsilon\|(\nabla v_\epsilon) d(x, \partial\Omega)^n\|_{L^2(\Omega)} \leq C(n)\epsilon^n. \tag{39}$$

In other words,

$$\sigma_{\text{boundary}} = \left\{\mu \in \mathbb{R}^+ \mid \exists(\mu_\epsilon, v_\epsilon) \textit{ solutions of } (37)\text{-}(38) \textit{ satisfying } (39)\right\}.$$

Remark that, compared to the Bloch spectrum, the definition of the boundary layer spectrum is not explicit. It may even depend on the choice of the sequence ϵ (on the contrary of σ_{Bloch}). There is also no guarantee that the boundary layer spectrum does not overlap the Bloch spectrum. From (39) we deduce that the sequence v_ϵ stays near the boundary $\partial\Omega$ at a maximum distance of the order of ϵ in the sense that

$$\lim_{\epsilon\to 0} \left(\|v_\epsilon\|_{L^2(\omega_\epsilon)} + \epsilon\|(\nabla v_\epsilon)\|_{L^2(\omega_\epsilon)}\right) = 0,$$

for any sequence of subsets ω_ϵ of Ω such that $d(\omega_\epsilon, \partial\Omega) >> \epsilon$. We shall say that such sequences of eigenvectors, whose limit eigenvalue belong to σ_{boundary}, *decrease exponentially fast* away from the boundary in the sense that, by virtue of (39), they decrease faster than any inverse power of the distance function $d(x, \partial\Omega)$.

The main result of this section states that, if the sequence of eigenvectors v_ϵ does not concentrate near the boundary, then automatically the limit eigenvalue μ belongs to the Bloch spectrum.

THEOREM 5.2. – *Let* v_ϵ *be a sequence of eigenvectors satisfying* (37) *and* (38). *If the limit eigenvalue* μ *does not belong to* σ_{boundary}, *then it must belong to* σ_{Bloch}. *Consequently, it implies*

$$\lim_{\epsilon\to 0} \epsilon^{-2}\sigma_\epsilon = \sigma_{\text{Bloch}} \cup \sigma_{\text{boundary}}.$$

REMARK 5.3. – *Theorem* 5.2 *gives only a sufficient condition (not necessary) for* μ *to belong to* σ_{Bloch}. *There may well be some limit eigenvalues* μ *which belong to both* σ_{Bloch} *and* σ_{boundary}. *We do not know if a more stringent definition of* σ_{boundary} *could yield an empty intersection between these two limit sets.*

Before proving Theorem 5.2, we introduce a definition of so-called "quasi eigenvectors" for the spectral problem (38) and prove several intermediate results.

DEFINITION 5.4. – *Let* μ_ϵ *be a sequence of eigenvalues for the spectral problem* (38) *which converges to a limit eigenvalue* μ. *A sequence* $u_\epsilon \in H^1(\mathbb{R}^N)$ *is said to be a sequence of* quasi eigenvectors *associated to the eigenvalues* μ_ϵ *if it satisfies*

(1) $u_\epsilon \equiv 0$ *in* $\mathbb{R}^N \setminus \Omega$ *and* $\|u_\epsilon\|_{L^2(\Omega)} = 1$,

(2) u_ϵ *is the solution in the sense of distributions in the whole space* $\mathbb{R}^N$ *of*

$$-\epsilon^2 \operatorname{div}\left[A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right] + u_\epsilon = \frac{1}{\mu_\epsilon}u_\epsilon + r_\epsilon, \tag{40}$$

where r_ϵ *is a remainder term which satisfies*

$$\lim_{\epsilon\to 0} \frac{\langle r_\epsilon, w_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)}}{\|w_\epsilon\|_{L^2(\mathbb{R}^N)} + \epsilon\|\nabla w_\epsilon\|_{L^2(\mathbb{R}^N)^N}} \dot{=} 0,$$

for all non-zero sequences $w_\epsilon \in H^1(\mathbb{R}^N)$.

REMARK 5.5. – *Equation* (40) *holds in* $\mathbb{R}^N$; *there is no more boundary conditions on* $\partial\Omega$. *Clearly, Definition* 5.4 *implies that a sequence of quasi eigenvectors* u_ϵ *satisfies also* $\epsilon\|\nabla u_\epsilon\|_{L^2(\mathbb{R}^N)^N} \le C$.

Sequences of quasi eigenvectors are easily built from sequences of eigenvectors which do not correspond to a limit eigenvalue $\mu \in \sigma_{\text{boundary}}$.

PROPOSITION 5.6. – *Let* v_ϵ *be a sequence of eigenvectors satisfying* (37) *and* (38). *Assume that it does not satisfy* (39), *namely that the limit eigenvalue does not belong to the boundary layer spectrum. Then, there exists a positive integer* $n \ge 1$ *and a subsequence, still denoted by* ϵ, *such that the sequence*

$$u_\epsilon = \frac{v_\epsilon d(x,\partial\Omega)^n}{\|v_\epsilon d(x,\partial\Omega)^n\|_{L^2(\Omega)}} \tag{41}$$

is a sequence of quasi-eigenvectors in the sense of Definition 5.1.

REMARK 5.7. – *We wrongly announced, in our previous note* [5], *that Proposition* 5.6 *is an alternative,* i.e. *that either a limit eigenvalue belongs to* σ_{boundary} *or there exists an associated sequence of quasi eigenvectors of the type given by* (41). *Unfortunately, we are unable to prove that, if there exists a sequence of quasi eigenvectors defined by* (41), *then the limit eigenvalue can not belong to* σ_{boundary}.

In several places in the sequel, the following estimate will often be used.

LEMMA 5.8. – *Let* v_ϵ *be a sequence of eigenvectors satisfying* (37) *and* (38). *For any positive integer* $n \ge 1$ *there exists a positive constant* $C(n) > 0$ *such that:*

$$\epsilon\|d(x,\partial\Omega)^n \nabla v_\epsilon\|_{L^2(\Omega)^N} \le C(n)\left[\|v_\epsilon d(x,\partial\Omega)^n\|_{L^2(\Omega)} + \epsilon\|v_\epsilon d(x,\partial\Omega)^{n-1}\|_{L^2(\Omega)}\right]$$

Proof. – For simplicity, let us denote by d the function $d(x, \partial\Omega)$ and by A^ϵ the matrix $A\left(x, \frac{x}{\epsilon}\right)$. Multiplying the spectral equation by $v_\epsilon d^{2n}$ leads to

$$\epsilon^2 \int_\Omega A^\epsilon(d^n \nabla v_\epsilon) \cdot (d^n \nabla v_\epsilon) = \left(\frac{1}{\mu_\epsilon} - 1\right) \int_\Omega (d^n u_\epsilon)^2 - 2n\epsilon^2 \int_\Omega v_\epsilon d^{2n-1} A^\epsilon \nabla v_\epsilon \cdot \nabla d. \tag{42}$$

Using the coercivity of A^ϵ in the left hand side, and estimating the right hand side, (42) yields

$$\epsilon^2 \|d^n \nabla v_\epsilon\|^2_{L^2(\Omega)^N} \leq C\Big(\|d^n v_\epsilon\|^2_{L^2(\Omega)} + \epsilon^2 \|d^n \nabla v_\epsilon\|_{L^2(\Omega)^N} \|d^{n-1} v_\epsilon\|_{L^2(\Omega)}\Big).$$

This gives the desired result.

Proof of Proposition 5.6. – If a sequence v_ϵ does not satisfy (39), then there exists a positive integer $n \geq 1$ and a subsequence, still denoted by ϵ, such that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \left(\|v_\epsilon d^n\|_{L^2(\Omega)} + \epsilon \|(\nabla v_\epsilon) d^n\|_{L^2(\Omega)^N}\right) = +\infty, \tag{43}$$

where, as in the previous proof, d denotes the function $d(x, \partial\Omega)$. Let us take the smallest integer n for which (43) holds. Necessarily $n \geq 1$ since $\|v_\epsilon\|_{L^2(\Omega)} + \epsilon\|\nabla v_\epsilon\|_{L^2(\Omega)^N}$ is bounded due to the spectral equation (38). Up to another subsequence, v_ϵ satisfies also

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^n} \|v_\epsilon d^n\|_{L^2(\Omega)} = +\infty. \tag{44}$$

Indeed, if it were not the case, $\epsilon^{-n}\|v_\epsilon d^n\|_{L^2(\Omega)}$ would be bounded while

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{n-1}} \|(\nabla v_\epsilon) d^n\|_{L^2(\Omega)^N} = +\infty.$$

By application of Lemma 5.8, this would imply that

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{n-1}} \|v_\epsilon d^{n-1}\|_{L^2(\Omega)} = +\infty,$$

which is a contradiction with our choice of n being the smallest integer such that (43) holds. Another consequence of such a choice is that for $n - 1$ there exists a constant C such that

$$\|v_\epsilon d^{n-1}\|_{L^2(\Omega)} + \epsilon \|(\nabla v_\epsilon) d^{n-1}\|_{L^2(\Omega)^N} \leq C\epsilon^{n-1}.$$

Let us define

$$u_\epsilon = \frac{v_\epsilon d^n}{\|v_\epsilon d^n\|_{L^2(\Omega)}}$$

and prove that u_ϵ is a sequence of quasi eigenvectors. Defining r_ϵ by equation (40), for any sequence of test functions $w_\epsilon \in H^1(\mathbb{R}^N)$ such that $\|w_\epsilon\|_{L^2(\Omega)} + \epsilon\|\nabla w_\epsilon\|_{L^2(\Omega)^N}$ is bounded uniformly in ϵ, we have

$$\langle r_\epsilon, w_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)} = \epsilon^2 \int_\Omega A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon \cdot \nabla w_\epsilon + \left(1 - \frac{1}{\mu_\epsilon}\right)\int_\Omega u_\epsilon w_\epsilon.$$

There is no contribution on the boundary $\partial\Omega$ because ∇u_ϵ (and not w_ϵ) vanishes on $\partial\Omega$ when $n \geq 1$. Using the spectral equation satisfied by v_ϵ, an integration by parts yields

$$\begin{aligned}
\langle r_\epsilon, w_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)} &= \frac{\epsilon^2}{\|v_\epsilon d^n\|_{L^2(\Omega)}}\left[\int_\Omega A^\epsilon \nabla(v_\epsilon d^n)\cdot\nabla w_\epsilon - \int_\Omega A^\epsilon\nabla v_\epsilon \cdot \nabla(d^n w_\epsilon)\right] \\
&= \frac{\epsilon^2}{\|v_\epsilon d^n\|_{L^2(\Omega)}}\int_\Omega n d^{n-1} A^\epsilon \nabla d\cdot(v_\epsilon \nabla w_\epsilon - w_\epsilon\nabla v_\epsilon) \\
&\leq C\epsilon \frac{\|v_\epsilon d^{n-1}\|_{L^2(\Omega)} + \epsilon\|d^{n-1}\nabla v_\epsilon\|_{L^2(\Omega)^N}}{\|v_\epsilon d^n\|_{L^2(\Omega)}} \\
&\leq C\frac{\epsilon^n}{\|v_\epsilon d^n\|_{L^2(\Omega)}},
\end{aligned}$$

which, by virtue of (44), goes to 0 with ϵ. Thus u_ϵ is a sequence of quasi eigenvectors.

This property of quasi eigenvectors can be localized in the sense that the matrix of coefficients $A\left(x, \frac{x}{\epsilon}\right)$ can be replaced by a purely periodically oscillating matrix $A(x_0, \frac{x}{\epsilon})$ for some fixed $x_0 \in \overline{\Omega}$.

PROPOSITION 5.9. – *Let u_ϵ be a sequence of quasi eigenvectors in the sense of Definition (5.4). Then, there exists a subsequence, still denoted by ϵ, a point $x_0 \in \overline{\Omega}$, and a sequence $\tilde{u}_\epsilon \in H^1(\mathbb{R}^N)$ of quasi eigenvectors for the matrix $A(x_0, \frac{x}{\epsilon})$,* i.e.:

(1) *$\tilde{u}_\epsilon \equiv 0$ in $\mathbb{R}^N \setminus \Omega$ and $\|\tilde{u}_\epsilon\|_{L^2(\Omega)} = 1$,*

(2) *$\tilde{u}_\epsilon$ is the solution in the sense of distributions in the whole space $\mathbb{R}^N$ of*

$$-\epsilon^2 \operatorname{div}\left[A(x_0, \frac{x}{\epsilon})\nabla\tilde{u}_\epsilon\right] + \tilde{u}_\epsilon = \frac{1}{\mu_\epsilon}\tilde{u}_\epsilon + \tilde{r}_\epsilon, \tag{45}$$

where $\tilde{r}_\epsilon$ is a remainder term which satisfies

$$\lim_{\epsilon\to 0}\frac{\langle \tilde{r}_\epsilon, w_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)}}{\|w_\epsilon\|_{L^2(\mathbb{R}^N)} + \epsilon\|\nabla w_\epsilon\|_{L^2(\mathbb{R}^N)^N}} = 0$$

for all non-zero sequences $w_\epsilon \in H^1(\mathbb{R}^N)$.

Proof. – If u_ϵ is a sequence of quasi eigenvectors, then there exists a sequence of real numbers α_ϵ converging to 0, such that

$$\left|\langle r_\epsilon, w_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)}\right| \leq \alpha_\epsilon\left(\|w_\epsilon\|_{L^2(\Omega)} + \epsilon\|\nabla w_\epsilon\|_{L^2(\Omega)^N}\right)$$

for any sequence $w_\epsilon \in H^1(\Omega)$ and with r_ϵ defined by equation (40). We introduce an intermediate scale $\beta_\epsilon > 0$ such that $\epsilon \ll \beta_\epsilon \ll 1$ and β_ϵ is an entire multiple of ϵ, *i.e.*

$$\lim_{\epsilon\to 0}\beta_\epsilon = 0, \quad \lim_{\epsilon\to 0}\frac{\epsilon}{\beta_\epsilon} = 0, \quad \text{and} \quad \frac{\beta_\epsilon}{\epsilon} = p_\epsilon \in \mathbb{N}.$$

The domain Ω is covered by a mesh of non-overlapping cubes $(P_i^\epsilon)_{1\leq i\leq n(\beta_\epsilon)}$ of the type $[0,\beta_\epsilon]^N$. The number of such cubes is $n(\beta_\epsilon)$, which is of the order of $\frac{|\Omega|}{\beta_\epsilon^N}$. We denote by x_i^ϵ the center of each cube P_i^ϵ, and by $i(\epsilon)$ the index such that the L^2-norm of u_ϵ is maximum on the cube $P_{i(\epsilon)}^\epsilon$. For the sake of simplicity, we denote $P_{i(\epsilon)}^\epsilon$ by P^ϵ. In other words,

$$\|u_\epsilon\|_{L^2(P^\epsilon)} = \max_{1\leq i\leq n(\beta_\epsilon)} \|u_\epsilon\|_{L^2(P_i^\epsilon)}. \tag{46}$$

Since $\sum_{1\leq i\leq n(\beta_\epsilon)} \|u_\epsilon\|^2_{L^2(P_i^\epsilon)} = 1$, we deduce that there exists a positive constant $C > 0$ such that

$$\|u_\epsilon\|_{L^2(P^\epsilon)} \geq C\beta_\epsilon^{N/2}. \tag{47}$$

Since $x_{i(\epsilon)}^\epsilon$ runs in the compact set $\overline{\Omega}$, there exists a subsequence, still denoted by ϵ, and a limit point $x_0 \in \overline{\Omega}$, such that

$$\lim_{\epsilon\to 0} x_{i(\epsilon)}^\epsilon = x_0.$$

Let us define a smooth function $\phi \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\begin{cases} \phi \geq 0 & \text{in} \quad \mathbb{R}^N, \\ \phi \equiv 1 & \text{in} \quad [-1/2, +1/2]^N, \\ \phi \equiv 0 & \text{outside} \quad [-1;+1]^N. \end{cases}$$

The quasi eigenvector u_ϵ is localized around $x_{i(\epsilon)}^\epsilon$ by multiplying it by a cut-off function ϕ_ϵ defined by

$$\phi_\epsilon(x) = \phi\left(\frac{x - x_{i(\epsilon)}^\epsilon}{\beta_\epsilon}\right). \tag{48}$$

This yields a function $\tilde{u}_\epsilon$ defined in $H^1(\mathbb{R}^N)$ by

$$\tilde{u}_\epsilon = \frac{\phi_\epsilon u_\epsilon}{\|\phi_\epsilon u_\epsilon\|_{L^2(\mathbb{R}^N)}}.$$

Let D^ϵ be the support of ϕ_ϵ. We choose the intermediate scale β_ϵ to be

$$\beta_\epsilon = \max\left(\sqrt{\epsilon}, \alpha_\epsilon^{1/N}\right). \tag{49}$$

(This implies that $\epsilon \ll \beta_\epsilon \ll 1$ and $\alpha_\epsilon^{2/N} \ll \beta_\epsilon$.) Then, by Lemma 5.10 below, the following estimates hold for u_ϵ in D^ϵ

$$\|u_\epsilon\|_{L^2(D^\epsilon)} \leq C\|u_\epsilon\|_{L^2(P^\epsilon)} \tag{50}$$

and

$$\epsilon\|\nabla u_\epsilon\|_{L^2(D^\epsilon)^N} \leq C\|u_\epsilon\|_{L^2(P^\epsilon)}. \tag{51}$$

To prove that $\tilde{u}_\epsilon$ is also a sequence of quasi eigenvectors for the matrix $A\left(x, \frac{x}{\epsilon}\right)$, we define a remainder term $\tilde{r}_\epsilon$ by

$$-\epsilon^2 \operatorname{div} A\left(x, \frac{x}{\epsilon}\right) \nabla \tilde{u}_\epsilon + \tilde{u}_\epsilon = \frac{1}{\mu_\epsilon} \tilde{u}_\epsilon + \tilde{r}_\epsilon. \tag{52}$$

We check the desired property for $\tilde{r}_\epsilon$ by multiplying 52) by $w_\epsilon \in H^1(\mathbb{R}^N)$. Integrating by parts and using equation (40) satisfied by u_ϵ, we obtain:

$$\begin{aligned}
\langle \tilde{r}_\epsilon, w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)} = & \frac{1}{\|\phi_\epsilon u_\epsilon\|_{L^2(\Omega)}} \left(\epsilon^2 \int_\Omega A^\epsilon \nabla (\phi_\epsilon u_\epsilon) \cdot \nabla w_\epsilon dx - \right. \\
& \left. - \epsilon^2 \int_\Omega A^\epsilon \nabla u_\epsilon \cdot \nabla (\phi_\epsilon w_\epsilon) + \langle r_\epsilon, \phi_\epsilon w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)} \right) \\
= & \frac{1}{\|\phi_\epsilon u_\epsilon\|_{L^2(\Omega)}} \left(\epsilon^2 \int_\Omega A^\epsilon \nabla \phi_\epsilon \cdot (u_\epsilon \nabla w_\epsilon - w_\epsilon \nabla u_\epsilon) dx + \right. \\
& \left. + \langle r_\epsilon, \phi_\epsilon w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
\left| \langle \tilde{r}_\epsilon, w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)} \right| \leq & \frac{C}{\|u_\epsilon\|_{L^2(P^\epsilon)}} \big(\epsilon^2 \beta_\epsilon^{-1} \big(\|u_\epsilon\|_{L^2(D^\epsilon)} \|\nabla w_\epsilon\|_{L^2(\Omega)^N} \\
& + \|w_\epsilon\|_{L^2(\Omega)} \|\nabla u_\epsilon\|_{L^2(D^\epsilon)^N} \big) \\
& + \alpha_\epsilon \big(\|\phi_\epsilon w_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla (\phi_\epsilon w_\epsilon)\|_{L^2(\Omega)^N} \big) \big).
\end{aligned}$$

Using estimates (47), (50), and (51) leads to

$$\begin{aligned}
\left| \langle \tilde{r}_\epsilon, w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)} \right| \leq & C \epsilon \beta_\epsilon^{-1} \big(\|w_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla w_\epsilon\|_{L^2(\Omega)^N} \big) \\
& + C \alpha_\epsilon \beta_\epsilon^{-N/2} \big(\|w_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla w_\epsilon\|_{L^2(\Omega)^N} \big).
\end{aligned}$$

From our choice (49) of β_ϵ, we know that both $\alpha_\epsilon \beta_\epsilon^{-N/2}$ and $\epsilon \beta_\epsilon^{-1}$ converge to 0, and we obtain the desired result

$$\lim_{\epsilon \to 0} \frac{\langle \tilde{r}_\epsilon, w_\epsilon \rangle_{H^{-1}, H^1(\mathbb{R}^N)}}{\|w_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla w_\epsilon\|_{L^2(\Omega)^N}} = 0. \tag{53}$$

To conclude, let us prove that $\tilde{u}_\epsilon$ is also a sequence of quasi eigenvectors for the matrix $A(x_0, \frac{x}{\epsilon})$. Defining a remainder term $\tilde{r}_\epsilon^0$ by

$$-\epsilon^2 \operatorname{div} A\left(x_0, \frac{x}{\epsilon}\right) \nabla \tilde{u}_\epsilon + \tilde{u}_\epsilon = \frac{1}{\mu_\epsilon} \tilde{u}_\epsilon + \tilde{r}_\epsilon^0;$$

we multiply it by $w_\epsilon \in H^1(\mathbb{R}^N)$ with $\|w_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla w_\epsilon\|_{L^2(\Omega)^N}$ uniformly bounded

$$\langle \tilde{r}_\epsilon^0, w_\epsilon \rangle = \langle \tilde{r}_\epsilon, w_\epsilon \rangle + \epsilon^2 \int_{D^\epsilon} \left(A\left(x_0, \frac{x}{\epsilon}\right) - A\left(x, \frac{x}{\epsilon}\right) \right) \nabla \tilde{u}_\epsilon \cdot \nabla w_\epsilon. \tag{54}$$

The first term in the right hand side of (54) goes to zero in view of (53), while the second term is bounded by

$$\left(\epsilon\|\nabla w_\epsilon\|_{L^2(\Omega)^N}\right)\left(\epsilon\|\nabla \tilde{u}_\epsilon\|_{L^2(\Omega)^N}\right)\sup_{x\in D^\epsilon}\|A(x_0,y)-A(x,y)\|_{L^\infty_\#(Y)^{N^2}},$$

which goes to zero with ϵ since the set D^ϵ concentrates near x_0 and the matrix $A(x,y)$ is continuous in x with values in $L^\infty_\#(Y)^{N^2}$.

LEMMA 5.10. – *Let P^ϵ be the cube of size β_ϵ where the L^2-norm of u_ϵ is maximum* (see (46)). *Let D^ϵ be the support of ϕ_ϵ* (see (48)). *If the intermediate scale β_ϵ is chosen such that*

$$\beta_\epsilon \geq \alpha_\epsilon^{2/N}, \tag{55}$$

then there exists a positive constant $C>0$ such that:

$$\|u_\epsilon\|_{L^2(D^\epsilon)} \leq C\|u_\epsilon\|_{L^2(P^\epsilon)} \tag{56}$$

and

$$\epsilon\|\nabla u_\epsilon\|_{L^2(D^\epsilon)^N} \leq C\|u_\epsilon\|_{L^2(P^\epsilon)}. \tag{57}$$

Proof. – Estimate (56) is obvious since P^ϵ is included in D^ϵ, which is covered by at most 3^N cubes P_i^ϵ, and the maximum L^2-norm is attained on P^ϵ. To prove estimate (57) a smooth function $\psi\in\mathcal{D}(\mathbb{R}^N)$ is introduced such that

$$\begin{cases}\psi\geq 0 & \text{in} \quad \mathbb{R}^N,\\ \psi\equiv 1 & \text{in} \quad [-1,+1]^N,\\ \psi\equiv 0 & \text{outside} \quad [-2;+2]^N.\end{cases}$$

Defining another cut-off function $\psi_\epsilon(x)=\psi(\frac{x-x_{i(\epsilon)}^\epsilon}{\beta_\epsilon})$ and multiplying equation (40), satisfied by u_ϵ, by $\psi_\epsilon^2 u_\epsilon$, yield

$$\begin{aligned}\epsilon^2\int_\Omega \psi_\epsilon^2 A^\epsilon\nabla u_\epsilon\cdot\nabla u_\epsilon &= \left(\frac{1}{\mu_\epsilon}-1\right)\int_\Omega \psi_\epsilon^2 u_\epsilon^2 dx\\ &- 2\epsilon^2\int_\Omega A^\epsilon(\psi_\epsilon\nabla u_\epsilon)\cdot(u_\epsilon\nabla\psi_\epsilon)+\langle r_\epsilon,\psi_\epsilon^2 u_\epsilon\rangle_{H^{-1},H^1(\mathbb{R}^N)}.\end{aligned} \tag{58}$$

Using the coercivity of A^ϵ in the left hand side, and estimating the right hand side, (58) leads to:

$$\begin{aligned}\epsilon^2\|\psi_\epsilon\nabla u_\epsilon\|^2_{L^2(\Omega)^N} &\leq C\Big(\epsilon^2\|\psi_\epsilon\nabla u_\epsilon\|_{L^2(\Omega)^N}\|u_\epsilon\nabla\psi_\epsilon\|_{L^2(\Omega)^N}+\|\psi_\epsilon u_\epsilon\|^2_{L^2(\Omega)}\\ &\quad+\alpha_\epsilon\left(\|\psi_\epsilon^2 u_\epsilon\|_{L^2(\Omega)}+\epsilon\|\nabla(\psi_\epsilon^2 u_\epsilon)\|_{L^2(\Omega)^N}\right)\Big)\\ &\leq C\Big(\epsilon^2\beta_\epsilon^{-1}\|\psi_\epsilon\nabla u_\epsilon\|_{L^2(\Omega)^N}\|u_\epsilon\|_{L^2(P^\epsilon)}+\|u_\epsilon\|^2_{L^2(P^\epsilon)}\\ &\quad+\alpha_\epsilon\left(\|u_\epsilon\|_{L^2(P^\epsilon)}+\epsilon\beta_\epsilon^{-1}\|u_\epsilon\|_{L^2(P^\epsilon)}+\epsilon\|\psi_\epsilon\nabla u_\epsilon\|_{L^2(\Omega)^N}\right)\Big)\\ &\leq C\Big(\epsilon\|\psi_\epsilon\nabla u_\epsilon\|_{L^2(\Omega)^N}\left(\alpha_\epsilon+\epsilon\beta_\epsilon^{-1}\|u_\epsilon\|_{L^2(P^\epsilon)}\right)+\|u_\epsilon\|^2_{L^2(P^\epsilon)}\\ &\quad+\alpha_\epsilon\|u_\epsilon\|_{L^2(P^\epsilon)}(1+\epsilon\beta_\epsilon^{-1})\Big),\end{aligned}$$

since the support of ψ_ϵ is covered by a finite number (independent of ϵ) of cubes P_i^ϵ and $\|\nabla\psi_\epsilon\|_{L^\infty(\mathbb{R}^N)^N} \leq C\beta_\epsilon^{-1}$. On the other hand, (47) and (55) implies that $\alpha_\epsilon \leq \beta_\epsilon^{N/2} \leq C\|u_\epsilon\|_{L^2(P^\epsilon)}$ and $\epsilon\beta_\epsilon^{-1}$ goes to zero. Finally, we obtain

$$\epsilon^2\|\psi_\epsilon\nabla u_\epsilon\|^2_{L^2(\Omega)^N} \leq C\Big(\|u_\epsilon\|^2_{L^2(P^\epsilon)} + \epsilon\|\psi_\epsilon\nabla u_\epsilon\|_{L^2(\Omega)^N}\|u_\epsilon\|_{L^2(P^\epsilon)}\Big),$$

from which the desired result (57) is easily deduced.

Proof of Theorem 5.2. – By Propositions 5.6 and 5.9 we already know that, if μ does not belong to σ_{boundary}, then there exist $x_0 \in \overline{\Omega}$ and a subsequence $\tilde{u}_\epsilon$ of quasi eigenvectors for the matrix $A(x_0, \frac{x}{\epsilon})$. Remark that if the matrix A depends only on y, and not on x, Proposition 5.9 is unnecessary since it is used only to "freeze" the macroscopic variable x. The quasi eigenvectors $\tilde{u}_\epsilon$ have compact support in $\overline{\Omega}$. Let us define K_ϵ as the smallest integer such that the cube $Q_\epsilon = [0; \epsilon K_\epsilon]^N$ contains Ω (K_ϵ is of the order of ϵ^{-1}). Since $\tilde{u}_\epsilon$ is identically equal to zero outside Ω, it belongs to $H^1_\#(Q_\epsilon)$.

In a **first step**, we apply the Bloch wave decomposition to $\tilde{u}_\epsilon$ in the cube Q_ϵ. Let j be a multi-index running in $\{0, 1, \ldots, K_\epsilon - 1\}^N$. For simplicity, we indicate its range by the notation $0 \leq j \leq K_\epsilon - 1$. According to Lemma 4.9 (and its generalization to Sobolev spaces), there exists a unique family $(u_\epsilon^j(y))_{0\leq j\leq K_\epsilon - 1}$ of functions in $H^1_\#(Y)$ such that

$$\tilde{u}_\epsilon(x) = \sum_{0\leq j\leq K_\epsilon-1} u_\epsilon^j\Big(\frac{x}{\epsilon}\Big)e^{2\pi i\frac{j\cdot x}{\epsilon K_\epsilon}}.$$

By Plancherel theorem,

$$\|\tilde{u}_\epsilon\|^2_{L^2(Q_\epsilon)} = (\epsilon K_\epsilon)^N \sum_{0\leq j\leq K_\epsilon-1} \|u_\epsilon^j\|^2_{L^2(Y)}$$

and

$$\epsilon^2\|\nabla\tilde{u}_\epsilon\|^2_{L^2(Q_\epsilon)^N} = (\epsilon K_\epsilon)^N \sum_{0\leq j\leq K_\epsilon-1} \Big\|\nabla_y u_\epsilon^j + 2i\pi\frac{j}{K_\epsilon}u_\epsilon^j\Big\|^2_{L^2(Y)^N}.$$

Remark that $(\epsilon K_\epsilon)^N$ is just the volume of the cube Q_ϵ and is therefore of order 1. Then, each Bloch component $u_\epsilon^j(y)$ is decomposed on the hilbertian basis of $L^2_\#(Y)$ made of the eigenfunctions of T_{θ,x_0} defined by (34), with the same Bloch frequency $\theta = j/K_\epsilon$. Here, x_0 is precisely the same point in $\overline{\Omega}$ that appear in the purely periodic matrix $A(x_0, \frac{x}{\epsilon})$. We denote by $(\mu^k(\theta), v^k(\theta, y))_{k\geq 1}$ the eigenvalues and eigenvectors of T_{θ,x_0} which satisfy $\|v^k\|_{L^2_\#(Y)} = 1$ and

$$(59) \qquad -\operatorname{div}_y\big[A(x_0, y)\nabla_y\big(v^k e^{2\pi i\theta\cdot y}\big)\big] + v^k e^{2\pi i\theta\cdot y} = \frac{1}{\mu^k(\theta)}v^k e^{2\pi i\theta\cdot y} \quad \text{in } Y.$$

Actually, the eigenvalues and eigenvectors $(\mu^k(\theta), v^k(\theta, y))_{k\geq 1}$ depend also on the point x_0. For simplicity, we do not state explicitly this dependence. There exist complex coefficients $\{\alpha_\epsilon^k(j/K_\epsilon)\}_{k\geq 1}$ such that

$$u_\epsilon^j(y) = \sum_{k\geq 1}\alpha_\epsilon^k\Big(\frac{j}{K_\epsilon}\Big)v^k\Big(\frac{j}{K_\epsilon}, y\Big).$$

The orthonormality property of the eigenfunctions implies that

$$\|u_\epsilon^j\|_{L^2(Y)}^2 = \sum_{k\geq 1} \left|\alpha_\epsilon^k(\frac{j}{K_\epsilon})\right|^2 .$$

In a **second step**, we introduce a *modulation* $\mathcal{M}(\tilde{u}_\epsilon)$ of the sequence of quasi-eigenvectors $\tilde{u}_\epsilon$ defined by:

$$\mathcal{M}(\tilde{u}_\epsilon)(x) = \sum_{0\leq j\leq K_\epsilon - 1} \sum_{k\geq 1} \psi^k\left(\frac{j}{K_\epsilon}\right)\alpha_\epsilon^k\left(\frac{j}{K_\epsilon}\right)v^k\left(\frac{j}{K_\epsilon}, \frac{x}{\epsilon}\right)e^{2\pi i \frac{j\cdot x}{\epsilon K_\epsilon}},$$

where the functions $\psi^k(\theta)$ are continuous, Y-periodic, and uniformly bounded

$$\sup_{k\geq 1} \|\psi^k\|_{C_\#(Y)} < +\infty.$$

It is easily seen that, by definition, the modulation $\mathcal{M}(\tilde{u}_\epsilon)$ belongs to $H^1_\#(Q_\epsilon)$ and satisfies the same a priori estimates than $\tilde{u}_\epsilon$

$$\|\mathcal{M}(\tilde{u}_\epsilon)\|_{L^2(Q_\epsilon)} + \epsilon\|\nabla\mathcal{M}(\tilde{u}_\epsilon)\|_{L^2(Q_\epsilon)^N} \leq C.$$

Multiplying the quasi-spectral equation (45) by the conjugate $\overline{\mathcal{M}(\tilde{u}_\epsilon)}$, leads to

$$\begin{aligned} &\epsilon^2 \int_{Q_\epsilon} A\left(x_0, \frac{x}{\epsilon}\right)\nabla\tilde{u}_\epsilon \cdot \nabla\overline{\mathcal{M}(\tilde{u}_\epsilon)}dx \\ &+ \left(1 - \frac{1}{\mu_\epsilon}\right)\int_{Q_\epsilon} \tilde{u}_\epsilon\overline{\mathcal{M}(\tilde{u}_\epsilon)}dx = \left\langle \tilde{r}_\epsilon^0, \overline{\mathcal{M}(\tilde{u}_\epsilon)}\right\rangle_{H^{-1},H_0^1(\mathbf{R}^N)}, \end{aligned} \tag{60}$$

in which the right hand side tends to 0 by virtue of Lemma 5.9. Then, using the orthogonality properties of the Bloch waves and of the eigenfunctions as well as the spectral equation (59), equation (60) becomes

$$\sum_{0\leq j\leq K_\epsilon - 1} \sum_{k\geq 1} \psi^k(\frac{j}{K_\epsilon})|\alpha_\epsilon^k(\frac{j}{K_\epsilon})|^2\left(\frac{1}{\mu^k(\frac{j}{K_\epsilon})} - \frac{1}{\mu_\epsilon}\right) = o(1), \tag{61}$$

where $o(1)$ tends to zero with ϵ.

In a **third step**, we define a family $\{\nu_\epsilon^k(\theta)\}_{k\geq 1}$ of *Bloch measures*, associated to the sequence $\tilde{u}_\epsilon$, by

$$\nu_\epsilon^k(\theta) = (\epsilon K_\epsilon)^N \sum_{0\leq j\leq K_\epsilon - 1} |\alpha_\epsilon^k(\frac{j}{K_\epsilon})|^2\delta_{\theta = \frac{j}{K_\epsilon}},$$

where $\delta_{\theta=\theta_0}$ denotes the Dirac mass at the frequency θ_0. Each $\nu_\epsilon^k(\theta)$ is a non-negative Radon measure defined in Y. Since $\tilde{u}_\epsilon$ has a unit norm in $L^2(\Omega)$, the sum of the integrals of these measures is equal to 1

$$\sum_{k\geq 1} \int_Y d\nu_\epsilon^k(\theta) = 1.$$

The sequence of Bloch measures is therefore bounded. Up to a subsequence, there exists a family of limit measures $\{\nu^k(\theta)\}_{k\geq 1}$ such that each ν_ϵ^k converges to ν^k in the sense of vague measures. Of course the limit measures are all non-negative. Let us show that they satisfy

$$\sum_{k\geq 1}\int_Y d\nu^k(\theta) = 1, \tag{62}$$

which proves that, at least, some of them are not identically zero. Of course, we have

$$0 \leq \sum_{k\geq 1}\int_Y d\nu^k(\theta) \leq 1.$$

For each fixed k we have

$$\lim_{\epsilon\to 0}\int_Y d\nu_\epsilon^k(\theta) = \int_Y d\nu^k(\theta).$$

If for all $\delta > 0$, there exists a rank k_δ such that, for any ϵ,

$$\sum_{k\geq k_\delta}\int_Y d\nu_\epsilon^k(\theta) \leq \delta,$$

then we easily deduce (62). Let us assume it is not the case: there exists a positive constant $\delta > 0$, a subsequence, still denoted by ϵ, and a sequence of integers $k(\epsilon)$, going to $+\infty$, such that

$$\sum_{k\geq k(\epsilon)}\int_Y d\nu_\epsilon^k(\theta) \geq \delta.$$

Now, recall that

$$\epsilon^2\|\nabla\tilde{u}_\epsilon\|^2_{L^2(Q_\epsilon)^N}$$
$$= (\epsilon K_\epsilon)^N \sum_{0\leq j\leq K_\epsilon-1}\sum_{k\geq 1}\left|\alpha_\epsilon^k\left(\frac{j}{K_\epsilon}\right)\right|^2 \|\nabla v^k\left(\frac{j}{K_\epsilon},y\right) + 2i\pi\frac{j}{K_\epsilon}v^k\left(\frac{j}{K_\epsilon},y\right)\|^2_{L^2(Y)^N}$$

and, by virtue of (59),

$$\|\nabla v^k(\theta,y) + 2i\pi\theta v^k(\theta,y)\|^2 \geq C\left(\frac{1}{\mu^k(\theta)} - 1\right) \quad \forall k, \forall\theta,$$

where the positive constant C does not depend on k nor θ. We deduce that

$$\epsilon^2\|\nabla\tilde{u}_\epsilon\|^2_{L^2(Q_\epsilon)^N} \geq C\delta\min_{\theta\in Y}\left(\frac{1}{\mu^{k(\epsilon)}(\theta)} - 1\right)$$

which goes to $+\infty$ since for any $\theta \in Y$

$$\lim_{k\to+\infty}\mu^k(\theta) = 0.$$

This is a contradiction with the fact that $\epsilon\nabla\tilde{u}_\epsilon$ is bounded in $L^2(\Omega)^N$. Therefore (62) is proved.

With the help of the Bloch measures, equation (61) can be rewritten

$$\sum_{k\geq 1}\int_Y \psi^k(\theta)\left(\frac{1}{\mu^k(\theta)}-\frac{1}{\mu_\epsilon}\right)d\nu_\epsilon^k(\theta)=o(1). \tag{63}$$

Since the test functions ψ^k and the eigenvalues μ^k are continuous in θ, one can pass to the limit in (63)

$$\sum_{k\geq 1}\int_Y \psi^k(\theta)\left(\frac{1}{\mu^k(\theta)}-\frac{1}{\mu}\right)d\nu^k(\theta)=0.$$

Since by virtue of (62) some of the limit measures ν^k are necessarily not zero, there exists at least one energy level k and a frequency θ such that

$$\mu=\mu^k(\theta),$$

which finishes the proof of Theorem 5.2.

REMARK 5.11. – *In the proof of Theorem* 5.2 *we used a sequence of quasi-eigenvectors* $\tilde{u}_\epsilon$ *rather than the sequence of true eigenvectors* v_ϵ. *The reason is that, although* $\tilde{u}_\epsilon$ *or* v_ϵ *are equal to 0 outside* Ω, *it is not the case for the modulation* $\mathcal{M}(\tilde{u}_\epsilon)$ *or* $\mathcal{M}(v_\epsilon)$. *Therefore, multiplying the spectral equation by* $\mathcal{M}(v_\epsilon)$ *and integrating by parts would produce a contribution on the boundary* $\partial\Omega$ *which, unfortunately, cannot be neglected. Such a difficulty does not occur with* $\tilde{u}_\epsilon$ *which satisfies an equation in* $\mathbb{R}^N$ *without boundary conditions* (see *Proposition* 5.9).

REMARK 5.12. – *We emphasize that the Bloch measure technique allows only to prove that*

$$\lim_{\epsilon\to 0}\sigma_\epsilon\subset\sigma_{\mathrm{Bloch}}\cup\sigma_{\mathrm{boundary}}.$$

The reverse inclusion has to be proved independently by the Bloch wave homogenization method. In this sense, these two methods are complementary.

The Bloch measures introduced here play, more or less, the role of semi-classical (or Wigner) measures in the context of Schrödinger equation (see e.g. [24], [31], [32], [47]).

6. Non-critical scalings

The goal of this section is to prove Theorem 3.2 concerning the asymptotic behavior of the rescaled spectrum $a_\epsilon^{-2}\sigma_\epsilon$ for a non critical scaling a_ϵ, *i.e.* for a sequence of positive numbers a_ϵ such that:

$$\lim_{\epsilon\to 0}a_\epsilon=0 \quad\text{and}\quad \begin{cases}\text{either} & \lim_{\epsilon\to 0}\frac{a_\epsilon}{\epsilon}=0,\\ \text{or} & \lim_{\epsilon\to 0}\frac{a_\epsilon}{\epsilon}=+\infty.\end{cases}$$

To study the eigenvalues λ_ϵ of (1) which are of the order of a_ϵ^2, we again modify slightly the spectral equation which becomes

$$(64)\qquad \begin{cases} -a_\epsilon^2 \operatorname{div}\left(A\left(x, \frac{x}{\epsilon}\right)\nabla v_\epsilon\right) + v_\epsilon = \frac{1}{\gamma_\epsilon} v_\epsilon & \text{in } \Omega \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

When labeling the eigenvalues of (1) and (64) in decreasing order, this has the effect of a change of variable for the eigenvalues

$$\gamma_\epsilon^k = \frac{\lambda_\epsilon^k}{a_\epsilon^2 + \lambda_\epsilon^k},$$

while leaving invariant the eigenfunctions v_ϵ^k. As in section 4, we introduce an operator $\tilde{S}_\epsilon \in \mathcal{L}(L^2(\Omega))$, associated to (64), defined by

$$(65)\qquad \begin{cases} \tilde{S}_\epsilon : L^2(\Omega) & \longrightarrow L^2(\Omega) \\ \qquad f & \longrightarrow u_\epsilon, \end{cases}$$

where u_ϵ is the unique solution in $H_0^1(\Omega)$ of

$$(66)\qquad \begin{cases} -a_\epsilon^2 \operatorname{div}\left[A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right] + u_\epsilon = f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

The analysis of the sequence of operators $\tilde{S}_\epsilon$ is similar to that presented in section 4. However, the main difference here is the absence of interaction between the homogenization scale ϵ and the singular perturbation scale a_ϵ. Roughly speaking, if ϵ is smaller than a_ϵ, then homogenization occurs first and the singular perturbation concerns the homogenized system. On the other hand, if ϵ is larger than a_ϵ, then the singular perturbation occurs first at a microscopic scale and homogenization is irrelevant. This yields some technical differences between these two cases. We begin with the largest scales a_ϵ.

6.1. Large scales: $\epsilon \ll a_\epsilon \ll 1$

As in section 4 we extend the operator $\tilde{S}_\epsilon$, originally defined in $L^2(\Omega)$, to a larger space of oscillating functions with period a_ϵ. For any positive number $\ell > 0$, let Z be the cube $Z = [0, \ell]^N$. We define an extended operator $S_\epsilon^\ell \in \mathcal{L}(L^2(\Omega \times Z))$ by

$$(67)\qquad S_\epsilon^\ell = E_\epsilon^\ell \tilde{S}_\epsilon P_\epsilon^\ell$$

where P_ϵ^ℓ and E_ϵ^ℓ are respectively a projection from $L^2(\Omega; L^2_\#(Z))$ into $L^2(\Omega)$ and an extension from $L^2(\Omega)$ into $L^2(\Omega; L^2_\#(Z))$. To insure that S_ϵ^ℓ is still self-adjoint, we ask P_ϵ^ℓ and E_ϵ^ℓ to be adjoint one from the other. To be sure that $\tilde{S}_\epsilon$ and S_ϵ^ℓ have the same spectrum, we ask the product $P_\epsilon^\ell E_\epsilon^\ell$ to be equal to the identity in $L^2(\Omega)$. Such conditions are satisfied by:

$$\begin{cases} \forall \phi(x,z) \in L^2(\Omega; L^2_\#(Z)),\ (P_\epsilon^\ell \phi)(x) = \sum_{i=1}^{n^\ell(\epsilon)} \chi_{Z_i^\epsilon}(x) \frac{1}{(\ell a_\epsilon)^N} \int_{Z_i^\epsilon} \phi(x', \frac{x}{a_\epsilon}) dx', \\ \forall f(x) \in L^2(\Omega),\ (E_\epsilon^\ell f)(x,z) = \sum_{i=1}^{n^\ell(\epsilon)} \chi_{Z_i^\epsilon}(x) f(x_i^\epsilon + a_\epsilon z), \end{cases}$$

where the family $(Z_i^\epsilon)_{1\leq i\leq n^\ell(\epsilon)}$ of non-overlapping cells of the type $[0;\ell a_\epsilon]^N$ covers Ω ($\chi_{Z_i^\epsilon}$ is the characteristic function of Z_i^ϵ and x_i^ϵ its origin). As before, S_ϵ^ℓ is self-adjoint because $(P_\epsilon^\ell)^* = E_\epsilon^\ell$ and its spectrum is exactly that of $\tilde{S}_\epsilon$ since $P_\epsilon^\ell E_\epsilon^\ell = Id_{L^2(\Omega)}$.

THEOREM 6.1. – *The sequence S_ϵ^ℓ converges strongly to a limit operator S^ℓ in the sense that, for any $\phi(x,z) \in L^2(\Omega; L^2_\#(Z))$, $S_\epsilon^\ell\phi$ converges strongly to $S^\ell\phi$ in $L^2(\Omega; L^2_\#(Z))$, and $S^\ell\phi = u^\ell$ is the unique solution in $L^2(\Omega; H^1_\#(Z))$ of*

$$-\operatorname{div}_z\left[A^*(x)\nabla_z u^\ell\right] + u^\ell = \phi \quad in \quad \Omega\times Z. \tag{68}$$

Moreover, S^ℓ is a self-adjoint non-compact operator in $L^2\big(\Omega; L^2_\#(Z)\big)$.

COROLLARY 6.2. – *For any choice of the sequence ϵ going to 0, the limit of $(a_\epsilon)^{-2}\sigma_\epsilon$ is the entire positive real axis, or equivalently*

$$\lim_{\epsilon\to\infty}\sigma(\tilde{S}_\epsilon) = \bigcup_{\ell>0}\sigma(S^\ell) = [0,1].$$

REMARK 6.3. – *Corollary 6.2 can be interpreted as a densification of the spectrum of S_ϵ upon rescaling at size a_ϵ^2. However, the limit problem (68) is probably not the only one to describe the limiting behavior of the spectrum at this range of frequency.*

Proof of Corollary 6.2. – From Theorem 6.1 we deduce that $\lim_{\epsilon\to 0}\sigma(S_\epsilon^\ell) \supset \sigma(S^\ell)$. Moreover, for any positive $\ell > 0$, the spectrum $\sigma(S^\ell)$ is obtained from $\sigma(S^1)$ by a simple transformation since the coefficient matrix in (6.8) does not depend on z. Labeling in increasing order the eigenvalues $(\gamma_\ell^k)_{k\geq 1}$ of S^ℓ, they satisfy

$$\gamma_\ell^k = \frac{\ell^2\gamma_1^k}{\ell^2\gamma_1^k + (1-\gamma_1^k)}.$$

By varying $\ell > 0$ the range of each eigenvalue γ_ℓ^k (for $k\geq 2$) is exactly $[0,1]$. Since all the eigenvalues of $\tilde{S}_\epsilon$ lie in $[0,1]$, this implies the desired result.

In order to prove Theorem 6.1, we have to analyze the asymptotic behavior of the sequence of solutions to problem (66) when $f = P_\epsilon^\ell[\phi(x,z)]$, where $\phi(x,z)$ is a given function in $L^2(\Omega; L^2_\#(Z))$. Problem (66) with such a sequence of right hand sides involves three different scales, namely 1, ϵ and a_ϵ. Therefore, the classical two-scale convergence is inoperative here, and one has to call for its generalization as described in [3]. We briefly recall the main results of the *multi-scale convergence method* (in our case, we just have three scales).

PROPOSITION 6.4. –

(1) *Let v_ϵ be a bounded sequence in $L^2(\Omega)$. There exist a subsequence, still denoted by ϵ, and a limit $v^0(x,z,y) \in L^2(\Omega; L^2_\#(Z\times Y))$ such that v_ϵ three-scale converges weakly to v^0 in the following sense*

$$\lim_{\epsilon\to 0}\int_\Omega v_\epsilon(x)\varphi(x,\frac{x}{a_\epsilon},\frac{x}{\epsilon})dx = \frac{1}{|Z\times Y|}\int_\Omega\int_{Z\times Y} v^0(x,z,y)\varphi(x,z,y)dxdzdy$$

for all functions $\varphi(x,z,y) \in L^2(\Omega; C_\#(Z\times Y))$.

(2) *Let v_ϵ be a sequence of functions in $L^2(\Omega)$ which three-scale converges weakly to a limit $v^0(x,y) \in L^2(\Omega; L^2_\#(Z \times Y))$. Assume furthermore that:*

$$\lim_{\epsilon\to 0} \|v_\epsilon\|^2_{L^2(\Omega)} = \frac{1}{|Z \times Y|}\|v^0\|^2_{L^2(\Omega;L^2_\#(Z\times Y))}.$$

Then v_ϵ is said to three-scale converge strongly *to v^0 in the sense that, for any sequence w_ϵ in $L^2(\Omega)$ which three-scale converges weakly to a limit $w^0(x,y) \in L^2(\Omega; L^2_\#(Z \times Y))$, we have*

$$\lim_{\epsilon\to 0}\int_\Omega v_\epsilon(x)w_\epsilon(x)\varphi(x,\frac{x}{a_\epsilon},\frac{x}{\epsilon})dx = \frac{1}{|Z \times Y|}\int_\Omega\int_{Z\times Y} v^0(x,z,y)w^0(x,z,y)\varphi(x,z,y)dxdzdy$$

for all smooth functions $\varphi(x,z,y) \in C(\overline{\Omega}; C_\#(Z \times Y))$.

(3) *Let v_ϵ be a bounded sequence in $L^2(\Omega)$ such that $a_\epsilon \nabla u_\epsilon$ is also bounded in $L^2(\Omega)^N$. There exist a subsequence, still denoted by ϵ, and a limit $v^0(x,z) \in L^2(\Omega; H^1_\#(Z))$, which is independent of y, such that v_ϵ three-scale converges to $v^0(x,z)$. Moreover, there exists $v^1(x,z,y) \in L^2(\Omega \times Z; H^1_\#(Y))$ such that $a_\epsilon \nabla u_\epsilon$ three-scale converges to $[\nabla_z v^0(x,z) + \nabla_y v^1(x,z,y)]$.*

Proof of Theorem 6.1. – Let $\phi(x,z)$ be a function in $L^2(\Omega\times Z)$ and $\theta^\epsilon(x,z)$ be a sequence of functions converging weakly to a limit $\theta(x,z)$ in $L^2(\Omega \times Z)$. By definition, we have

$$\langle S^\ell_\epsilon \phi(x,z), \theta^\epsilon(x,z)\rangle = \int_\Omega u^\epsilon(x)\big(P^\ell_\epsilon \theta^\epsilon\big)(x)dx,$$

where u_ϵ is the solution of

$$\begin{cases} -a_\epsilon^2 \operatorname{div} A\Big(x, \frac{x}{\epsilon}\Big)\nabla u_\epsilon + u_\epsilon = \big(P^\ell_\epsilon \phi\big)(x) & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{69}$$

From Lemma 4.7, we know that $(P^\ell_\epsilon \theta^\epsilon)(x)$ two-scale converges to $\theta(x,z)$. From Lemma 6.5 below, $u_\epsilon(x)$ two-scale converges strongly to $u^0(x,z)$, solution of (70). Thus, defining a limit operator S^ℓ by $u^0 = S^\ell\phi$, we obtain

$$\lim_{\epsilon\to 0}\langle S^\ell_\epsilon \phi(x,z), \theta^\epsilon(x,z)\rangle = \frac{1}{|Z|}\int_\Omega\int_Z u^0(x,z)\theta(x,z)dxdz = \langle S^\ell\phi, \theta\rangle.$$

Therefore, the sequence of operators S^ℓ_ϵ converges strongly to S^ℓ, and as a consequence

$$\lim_{\epsilon\to 0}\sigma(S^\ell_\epsilon) \supset \sigma(S^\ell).$$

LEMMA 6.5. – *Let u_ϵ be the solution of problem (69). The sequence $u_\epsilon(x)$ two-scale converges strongly to $u^0(x,z)$, where u^0 is the unique solution in the space $L^2(\Omega; H^1_\#(Z))$ of*

$$\left\{\begin{array}{l} -\operatorname{div}_z[A^*(x)\nabla_z u^0] + u^0 = \phi \quad \text{in } \Omega\times Z \\ z \to u^0(x,z) \ Z\text{-periodic.}\end{array}\right. \tag{70}$$

Proof. – Standard *a priori* estimates on u_ϵ are

$$\|u_\epsilon\|_{L^2(\Omega)} + a_\epsilon\|\nabla u_\epsilon\|_{L^2(\Omega)^N} \le C.$$

By virtue of Proposition 6.4, there exists a subsequence, still denoted by ϵ, and two limits $u^0(x,z) \in L^2(\Omega; H^1_\#(Z))$ and $v^1(x,z,y) \in L^2(\Omega; H^1_\#(Z\times Y))$, such that u_ϵ and $a_\epsilon \nabla u_\epsilon$ three-scale converge weakly to u^0 and $\nabla_z u^0 + \nabla_y u^1$ respectively. Let us now multiply equation (69) by a test function of the following form

$$\psi\left(x, \frac{x}{a_\epsilon}\right) + \frac{\epsilon}{a_\epsilon}\psi^1\left(x, \frac{x}{a_\epsilon}, \frac{x}{\epsilon}\right),$$

where $\psi(x,z)$ and $\psi^1(x,z,y)$ are smooth functions, periodic in z and y. Integrating by parts, and recalling that $a_\epsilon \gg \epsilon$, we get

$$\int_\Omega A\left(x, \frac{x}{\epsilon}\right) a_\epsilon \nabla u_\epsilon \cdot [\nabla_z\psi + \nabla_y\psi^1]\left(x, \frac{x}{a_\epsilon}, \frac{x}{\epsilon}\right) dx$$
$$+ \int_\Omega u_\epsilon \psi(x, \frac{x}{a_\epsilon}) dx = \int_\Omega (P^\ell_\epsilon \phi)(x)\psi\left(x, \frac{x}{a_\epsilon}\right) dx + o(1).$$

Passing to the limit, it becomes:

$$\int_\Omega\int_{Z\times Y} A(x,y)[\nabla_z u^0 + \nabla_y u^1]\cdot[\nabla_z\psi + \nabla_y\psi^1] dx dy dz$$
$$+ \int_\Omega\int_{Z\times Y} u^0\psi dx dy dz = \int_\Omega\int_{Z\times Y} \phi(x,z)\psi(x,z) dx dy dz.$$

Since u^0 and ϕ are independent of y, the y variable can be eliminated by introducing the solutions of the usual local problems

$$\left\{\begin{array}{l} -\operatorname{div}_y A(x,y)(\nabla_y w^i(x,y) + e_i) = 0 \quad \text{in } Y\times\Omega \\ y \mapsto w^i(x,y) \ Y\text{-periodic.}\end{array}\right.$$

A simple calculation shows that

$$u^1(x,y,z) = \sum_{i=1}^N \frac{\partial u^0}{\partial z_i}(x,z) w^i(x,y).$$

Thus, $u^0(x,z)$ resolves the homogenized problem corresponding to the usual homogenized matrix $A^*(x)$.

To prove that u_ϵ two-scale converges strongly to u^0, we repeat the argument used in the proof of Lemma 4.8. Thanks to Lemma 4.7, applied to the reference cell Z instead of KY, we see that $P^\ell_\epsilon\phi(x,z)$ two-scale converges strongly to $\phi(x,z)$. Therefore, we have

$$\lim_{\epsilon\to 0}\int_\Omega \left(a_\epsilon^2 A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\cdot\nabla u_\epsilon + u_\epsilon^2\right) dx = \frac{1}{|Z|}\int_\Omega\int_Z \left(A^*(x)\nabla_z u^0\cdot\nabla_z u^0 + (u^0)^2\right) dx dz.$$

Therefore, using the lower semi-continuity of the two-scale convergence (*see* [2]) we conclude that u_ϵ two-scale converges strongly to u^0.

6.2. Small scales: $a_\epsilon \ll \epsilon$

To study the case of small scales, we keep the definition (65-66) of the operator $\tilde{S}_\epsilon$ and slightly modify the extension S_ϵ^ℓ as follows

$$S_\epsilon^\ell = E_\epsilon^\ell \tilde{S}_\epsilon P_\epsilon^\ell \ : \ L^2(\Omega \times Y \times Z) \longrightarrow L^2(\Omega \times Y \times Z),$$

where $Z = [0, \ell]^N$ is the reference cell and P_ϵ^ℓ is a new projection operator, defined by

$$P_\epsilon^\ell = P_\epsilon^{(1)} P_\epsilon^{(2)} \ : \ L^2(\Omega \times Y \times Z) \longrightarrow L^2(\Omega),$$

where $P_\epsilon^{(1)}$ and $P_\epsilon^{(2)}$ are respectively a projection operator from $L^2(\Omega \times Y)$ onto $L^2(\Omega)$ and a projection from $L^2(\Omega \times Y \times Z)$ onto $L^2(\Omega \times Y)$. They are defined as follows

$$\forall \phi(x,y) \in L^2(\Omega \times Y), \quad \left(P_\epsilon^{(1)} \phi\right)(x) = \frac{1}{\epsilon^N} \int_{Y_i^\epsilon} \phi(x', \frac{x}{\epsilon}) dx'$$

in each one of the non-overlapping cells Y_i^ϵ of the type $[0; \epsilon]^N$ covering Ω, and

$$\forall \phi(x,y,z) \in L^2(\Omega \times Y \times Z), \quad \left(P_\epsilon^{(2)} \phi\right)(x,y) = (\frac{\epsilon}{\ell a_\epsilon})^N \int_{\tilde{Z}_i^\epsilon} \phi(x, y', \frac{\epsilon y}{a_\epsilon}) dy'$$

in each one of the non-overlapping cubes Z_i^ϵ of the type $[0, \frac{\ell a_\epsilon}{\epsilon}]^N$ covering Y.

On the other hand, E_ϵ^ℓ is an extension operator whose definition is as follows:

$$E_\epsilon^\ell = E_\epsilon^{(2)} E_\epsilon^{(1)} \ : \ L^2(\Omega) \longrightarrow L^2(\Omega \times Y \times Z),$$

where, $E_\epsilon^{(2)}$ and $E_\epsilon^{(1)}$ are respectively an extension operator from $L^2(\Omega)$ into $L^2(\Omega \times Y)$ and an extension from $L^2(\Omega \times Y)$ into $L^2(\Omega \times Y \times Z)$. They are defined by:

$$\forall f(x) \in L^2(\Omega), \quad (E_\epsilon^{(2)} f)(x,y) = \sum_{i=1}^{n(\epsilon)} \chi_i^\epsilon(x) f(x_i^\epsilon + \epsilon y),$$

where, as usual, x_i^ϵ is the origin of each cell Y_i^ϵ and χ_i^ϵ is its characteristic function and

$$\forall \phi(x,y) \in L^2(\Omega \times Y), \quad (E_\epsilon^{(1)} \phi)(x,y,z) = \sum_{i=1}^{n^\ell(\epsilon)} \chi_{Z_i^\epsilon}(x) \phi(x_i^\epsilon + \epsilon y, y_i^\epsilon + \frac{a_\epsilon \ell}{\epsilon} z),$$

where y_i^ϵ is the origin of each cell Z_i^ϵ.

It can be checked that $E_\epsilon^\ell = (P_\epsilon^\ell)^*$ and $P_\epsilon^\ell E_\epsilon^\ell = Id_{L^2(\Omega)}$. Therefore, S_ϵ^ℓ is also a self-adjoint, compact operator and it has the same spectrum as $\tilde{S}_\epsilon$.

THEOREM 6.6. – *The sequence of operators S_ϵ^ℓ converges strongly to a limit S^ℓ which is given by*

$$\begin{cases} S^\ell : L^2(\Omega \times Y \times Z) & \longrightarrow L^2(\Omega \times Y \times Z) \\ \qquad \phi(x,y,z) & \longmapsto u^0(x,y,z), \end{cases}$$

where $u^0(x,y,z)$ *is the unique solution in* $L^2(\Omega \times Y; H^1_\#(Z))$ *of*

$$\begin{cases} -\operatorname{div}_z A(x,y)\nabla_z u^0 + u^0 = \phi & \text{in} \quad \Omega \times Y \times Z \\ z \longmapsto u^0 \quad Z\text{-periodic.} \end{cases}$$

Therefore, $\lim_{\epsilon \to 0} \sigma(S^\ell_\epsilon) \supset \sigma(S^\ell)$ *and furthermore,*

$$\lim_{\epsilon \to 0} \sigma(\tilde{S}_\epsilon) = \bigcup_{\ell > 0} \sigma(S^\ell) = [0,1].$$

Proof. – The proof is completely analogous to that of Theorem 6.1. It appeals to the same ingredients, and still uses techniques from the three-scale convergence method. In particular, a new technical lemma generalizing Lemma 4.7 is required in order to prove that:

(1) If $\theta(x,y,z)$ is a given function in $L^2(\Omega; L^2_\#(Y \times Z))$, then the sequence $(P^\ell_\epsilon \theta)(x)$ converges strongly in the sense of three-scale convergence to $\theta(x,y,z)$.

(2) If $\theta_\epsilon(x,y,z)$ is a sequence converging weakly to $\theta(x,y,z)$ in $L^2(\Omega; L^2_\#(Y \times Z))$, then the sequence $(P^\ell_\epsilon \theta_\epsilon)(x)$ converges weakly in the sense of three-scale convergence to $\theta(x,y,z)$.

Since these results are quite standard generalizations of their two-scale counterparts, we refer to Lemma 4.7 without further details.

7. Boundary layer spectrum

The goal of this section is to prove Theorem 3.5 which characterizes the boundary layer spectrum. We proceed in two steps corresponding to sections 4 and 5 adapted to the special case of σ_{boundary}. In a first subsection, we extend the operators $\tilde{S}_\epsilon$ to a functional space made of functions which oscillate transversaly to a plane boundary Σ and which decay away from Σ. This extended sequence of operators converges to a new limit operator which captures these sequences of eigenvectors concentrating on Σ. We characterize this limit spectrum σ_Σ which may contain new eigenvalues not included in σ_{Bloch}. In a second subsection, we prove a completeness result which states that the boundary layer spectrum σ_{boundary} is precisely contained in the union of all the limit spectra σ_Σ corresponding to the different parts Σ that make up the whole boundary $\partial\Omega$. Finally, a third subsection is devoted to a brief generalization of the previous analysis to the case when Σ is a lower-dimensional part of the boundary $\partial\Omega$, namely corners in 2-D.

7.1. Boundary layer homogenization

In this subsection we assume that Ω is a cylindrical bounded open set in $\mathbb{R}^N$ in the sense that there exist Σ, a bounded open set in $\mathbb{R}^{N-1}$, and $L > 0$, a positive length, such that

$$\Omega = \Sigma \times]0; L[. \tag{71}$$

With no loss of generality, we assume that the axis of the cylindrical domain Ω is the N^{th} direction: a generic point $x \in \Omega$ is denoted by $x = (x', x_N)$ with $x' \in \Sigma$ and $x_N \in]0; L[$.

The goal of this subsection is to analyze the asymptotic behavior of that part of the spectrum $\sigma(\tilde{S}_\epsilon)$ which corresponds to eigenvectors concentrating on the boundary $\Sigma \times \{0\}$. At this point, no restrictions are made on the sequence ϵ which goes to zero.

Similarly, we define a semi-infinite band

$$G = Y' \times]0; +\infty[,$$

where $Y' =]0, 1[^{N-1}$ is the unit cell in $\mathbb{R}^{N-1}$. A generic point y in G is denoted by $y = (y', y_N)$ with $y' \in Y'$ and $y_N \in \mathbb{R}^+$.

Recall that the operator $\tilde{S}_\epsilon$ is defined by (*see* (23)):

$$\left\{ \begin{array}{lcl} \tilde{S}_\epsilon : L^2(\Omega) & \longrightarrow & L^2(\Omega) \\ f & \longrightarrow & u_\epsilon, \end{array} \right. \tag{72}$$

where u_ϵ is the unique solution in $H_0^1(\Omega)$ of

$$\left\{ \begin{array}{ll} -\epsilon^2 \operatorname{div}\left[A(x, \frac{x}{\epsilon}) \nabla u_\epsilon\right] + u_\epsilon = f & \text{in } \Omega, \\ u_\epsilon = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{73}$$

As in section 4, we extend the operator $\tilde{S}_\epsilon$ to a space of two-scale oscillating functions. However, here we choose a space corresponding to boundary layers near $\Sigma \times \{0\}$: $L^2(\Sigma; L^2_\#(KG))$ where $K \geq 1$ is a given positive integer, and KG denotes the semi-infinite band $[0, K]^{N-1} \times]0; +\infty[$. It is a space of two-scale functions oscillating periodically in y' parallel to Σ, and decaying to 0 as y_N goes to infinity (in the sense of square integrable functions in the semi-infinite band G). More precisely, $L^2_\#(G)$ is defined by:

$$L^2_\#(G) = \{\phi(y) \in L^2(G) \mid y' \mapsto \phi(y', y_N) \text{ is } Y'\text{-periodic}\}.$$

Similarly, we define $H^1_{0\#}(G)$ as the Sobolev space of all functions in $H^1(G)$ which are Y'-periodic and vanishes for $y_N = 0$

$$H^1_{0\#}(G) = \{\phi(y) \in H^1(G) \mid y' \mapsto \phi(y', y_N) \text{ is } Y'\text{-periodic, and } \phi(y', 0) = 0\}. \tag{74}$$

An extented operator $B^K_\epsilon \in \mathcal{L}(L^2(\Sigma; L^2_\#(KG)))$ is defined by

$$B^K_\epsilon = E^K_\epsilon \tilde{S}_\epsilon P^K_\epsilon, \tag{75}$$

where P^K_ϵ and E^K_ϵ are respectively a projection from $L^2(\Sigma; L^2_\#(KG))$ onto $L^2(\Omega)$ and an extension from $L^2(\Omega)$ into $L^2(\Sigma; L^2_\#(KG))$. To insure that B^K_ϵ is still self-adjoint, we ask P^K_ϵ and E^K_ϵ to be adjoint one from the other. To be sure that $\tilde{S}_\epsilon$ and B^K_ϵ have the same spectrum, we ask the product $P^K_\epsilon E^K_\epsilon$ to be equal to the identity in $L^2(\Omega)$. The Hilbert space $L^2(\Sigma; L^2_\#(KG))$ is equipped with the scalar product

$$\langle \phi, \psi \rangle = \frac{1}{K^{N-1}} \int_\Sigma \int_{KG} \phi(x', y) \psi(x', y) dx' dy.$$

To build such extension and projection operators, we introduce a regular mesh of size $K\epsilon$ of the boundary Σ: let $(\Sigma_i^\epsilon)_{1 \leq i \leq n'(\epsilon)}$ be a family of non-overlapping cells of the type

$[0;K\epsilon]^{N-1}$ covering Σ (the number of cells is $n'(\epsilon)$ which is of the order of $\frac{|\Sigma|}{(K\epsilon)^{N-1}}$). We denote by $x_i'^\epsilon$ the origin of each cell Σ_i^ϵ, and by $\chi_i^\epsilon(x')$ its characteristic function. Defining a projection operator by:

$$P_\epsilon^K : L^2(\Sigma; L^2_\#(KG)) \longrightarrow L^2(\Omega)$$

$$\phi(x',y) \longmapsto \sum_{i=1}^{n'(\epsilon)} \chi_i^\epsilon(x') \frac{1}{(K\epsilon)^{N-1}} \int_{\Sigma_i^\epsilon} \phi(z',\frac{x}{\epsilon})dz', \tag{76}$$

and an extension operator by

$$E_\epsilon^K : L^2(\Omega) \longrightarrow L^2(\Sigma; L^2_\#(KG))$$

$$f(x) \longmapsto \sum_{i=1}^{n'(\epsilon)} \chi_i^\epsilon(x') f(x_i'^\epsilon + \epsilon y', \epsilon y_N), \tag{77}$$

their announced properties are checked in the following

LEMMA 7.1. – *The operators P_ϵ^K and E_ϵ^K defined by* (76) *and* (77) *satisfy:*

$$\|E_\epsilon^K f\|_{L^2(\Sigma;L^2_\#(KG))} \le \frac{C}{\sqrt{\epsilon}} \|f\|_{L^2(\Omega)}, \quad \|P_\epsilon^K \phi\|_{L^2(\Omega)} \le C\sqrt{\epsilon}\|\phi\|_{L^2(\Sigma;L^2_\#(KG))}$$

and

$$P_\epsilon^K E_\epsilon^K = \mathrm{Id}_{L^2(\Omega)}, \quad \left(P_\epsilon^K\right)^* = \epsilon E_\epsilon^K.$$

Furthermore, the product $E_\epsilon^K P_\epsilon^K$ converges strongly to the identity in the space $\mathcal{L}\big(L^2(\Sigma; L^2_\#(KG))\big)$.

Proof. – The proof of lemma (7.1) is very similar to that of Lemma 4.2. Therefore, we simply sketch the derivation of the estimate for $P_\epsilon^K\phi$ (that for $E_\epsilon^K f$ is parallel). By definition of the mesh $(\Sigma_i^\epsilon)_{1\le i\le n'(\epsilon)}$, we have

$$\int_\Omega |P_\epsilon^K \phi|^2 dx = \sum_{i=1}^{n'(\epsilon)} \int_{\Sigma_i^\epsilon \times]0,L[} |P_\epsilon^K \phi|^2 dx.$$

Since by Cauchy-Schwartz inequality in Σ_i^ϵ

$$|P_\epsilon^K \phi|^2 \le \frac{1}{(K\epsilon)^{N-1}} \int_{\Sigma_i^\epsilon} |\phi\left(z', \frac{x}{\epsilon}\right)|^2 dz',$$

we deduce

$$\int_\Omega |P_\epsilon^K \phi|^2 dx \le \frac{1}{(K\epsilon)^{N-1}} \sum_{i=1}^{n'(\epsilon)} \int_{\Sigma_i^\epsilon} \int_{\Sigma_i^\epsilon \times]0,L[} |\phi\left(z', \frac{x}{\epsilon}\right)|^2 dx dz'.$$

By the change of variables $y = \frac{x}{\epsilon}$, we obtain

$$\int_\Omega |P_\epsilon^K \phi|^2 dx \le C\epsilon \sum_{i=1}^{n'(\epsilon)} \int_{\Sigma_i^\epsilon} \int_{KY' \times \mathbb{R}^+} |\phi(z', y)|^2 dy dz'$$
$$\le C\epsilon \int_\Sigma \int_{KY' \times \mathbb{R}^+} |\phi(z', y)|^2 dy dz' = C\epsilon \|\phi\|^2_{L^2(\Sigma; L^2_\#(KG))}.$$

THEOREM 7.2. – *The sequence B_ϵ^K converges strongly in $\mathcal{L}(L^2(\Sigma; L^2_\#(KG)))$ to a limit operator B^K defined, for any $\phi \in L^2(\Sigma; L^2_\#(KG))$, by $B^K\phi = u^K$ the unique solution in $L^2(\Sigma; H^1_{0\#}(KG))$ of:*

$$(78) \qquad \begin{cases} -\operatorname{div}_y\left[A((x',0),y)\nabla_y u^K(x',y)\right] + u^K(x',y) = \phi(x',y) & \text{in } \Sigma \times KG \\ u^K(x',(y',0)) = 0 & \text{on } y_N = 0 \\ y' \to u^K(x',(y',y_N)) \ [0,K]^{N-1}\text{-periodic.} & \end{cases}$$

Moreover, B^K is a self-adjoint non-compact operator in $\mathcal{L}(L^2(\Sigma; L^2_\#(KG)))$ satisfying

$$\sigma(B^K) \subset \lim_{\epsilon \to 0} \sigma(\tilde{S}_\epsilon).$$

Remark that, since the solution u^K of (78), considered as a function of y, belongs to $H^1_{0\#}(KG)$, it decreases to 0 in a weak sense as y_N goes to infinity. Of course, by Rellich theorem (*see e.g.* [42], [28], [43]), we can also deduce from the strong convergence of B_ϵ^K to B^K the corresponding strong convergence of the spectral families which can be interpreted as an "averaged" convergence of the eigenvectors. To compute the spectrum of B^K, we diagonalize B^K by using a variant of the Bloch wave decomposition: Bloch frequencies are introduced only for the space variables parallel to the boundary Σ. The proof of this partial Bloch wave decomposition is identical to that of Lemma 4.9.

LEMMA 7.3. – *For any function $\phi(y) \in L^2_\#(KG)$ there exists a unique family $\{\phi_{j'}(y)\} \in L^2_\#(G)^{K^{N-1}}$, indexed by a multi-index j' whose $N-1$ components belong to $\{0, \ldots, K-1\}$, such that*

$$\phi(y) = \sum_{0 \le j' \le K-1} \phi_{j'}(y) e^{2\pi i \frac{j' \cdot y'}{K}}$$

and

$$\frac{1}{K^{N-1}} \int_{KG} |\phi|^2 dy = \sum_{0 \le j' \le K-1} \int_G |\phi_{j'}|^2 dy.$$

This decomposition, denoted by $\mathcal{B}'$, defines a unitary isometry from $L^2_\#(KG)$ into $L^2_\#(G)^{K^{N-1}}$

From Lemma 7.3, we easily deduce the following:

PROPOSITION 7.4. – *The operator B^K can be diagonalized as*

$$B^K = \mathcal{B}'^* \operatorname{diag}\left[(B_{j'/K})_{0 \le j' \le K-1}\right] \mathcal{B}',$$

where, for each Bloch frequency $\theta' = j'/K$, $B_{\theta'}$ *is a self-adjoint non-compact operator defined in* $\mathcal{L}(L^2(\Sigma; L^2_\#(G)))$ *by*

$$\left\{\begin{array}{lll} B_{\theta'} : L^2(\Sigma; L^2_\#(G)) & \longrightarrow & L^2(\Sigma; L^2_\#(G)) \\ \phi & \longrightarrow & u, \end{array}\right. \tag{79}$$

where $u(x', y)$ *is the unique solution in* $L^2(\Sigma; H^1_{0\#}(G))$ *of*

$$\left\{\begin{array}{ll} -\operatorname{div}_y\Big[A((x',0),y)\nabla_y\Big(ue^{2\pi i\theta'\cdot y'}\Big)\Big] + ue^{2\pi i\theta'\cdot y'} = \phi e^{2\pi i\theta'\cdot y'} & \textit{in } \Sigma\times G \\ u(x',(y',0)) = 0 & \textit{on } y_N = 0 \\ y' \to u(x',(y',y_N)) \ Y'\textit{-periodic.} & \end{array}\right.$$

The spectrum of B^K *is then* $\sigma(B^K) = \bigcup_{0\le j'\le K-1} \sigma(B_{\frac{j'}{K}})$.

In order to characterize the spectrum of $B_{\theta'}$, we freeze the x' variable. For any fixed $x' \in \overline{\Sigma}$ and $\theta' \in Y'$ we introduce an operator $B_{\theta',x'}$ acting on $L^2_\#(G)$, defined by

$$\left\{\begin{array}{lll} B_{\theta',x'} : L^2_\#(G) & \longrightarrow & L^2_\#(G) \\ \phi & \longrightarrow & u, \end{array}\right. \tag{80}$$

where $u(y)$ is the unique solution in $H^1_{0\#}(G)$ (defined by 74)) of:

$$\left\{\begin{array}{ll} -\operatorname{div}_y\Big[A((x',0),y)\nabla_y\Big(ue^{2\pi i\theta'\cdot y'}\Big)\Big] + ue^{2\pi i\theta'\cdot y'} = \phi e^{2\pi i\theta'\cdot y'} & \text{in } G \\ u(y',0) = 0 & \text{on } y_N = 0 \\ y' \to u(y',y_N) \ Y'\text{-periodic.} & \end{array}\right.$$

Remark that each operator $B_{\theta',x'}$ is non-compact since the band G is unbounded. Therefore its spectrum may be not purely discrete. As usual its spectrum can be decomposed into its discrete and essential parts

$$\sigma(B_{\theta',x'}) = \sigma_{disc}(B_{\theta',x'}) \cup \sigma_{ess}(B_{\theta',x'}).$$

PROPOSITION 7.5. – *The operator* $B_{\theta',x'}$ *is a self-adjoint non-compact operator whose essential spectrum is given by*

$$\sigma_{ess}(B_{\theta',x'}) = \bigcup_{0\le\theta_N\le 1} \sigma(T_{(\theta',\theta_N),(x',0)}),$$

where $T_{\theta,x}$ *is the operator defined in Proposition* 4.12. *Furthermore, each discrete eigenvalue in* $\sigma_{disc}(B_{\theta',x'})$ *is locally continuous in* (θ', x'), *and its associated eigenvector is exponentially decreasing when* y_N *goes to infinity.*

Globally, the spectrum of $B_{\theta',x'}$ *is continuous in* (θ', x') *as a subset of* $\mathbb{R}^+$. *Therefore, defining the limit spectrum associated to the surface* Σ *by*

$$\sigma_\Sigma = \lim_{K\to+\infty} \sigma(B^K), \tag{81}$$

we have

$$\sigma_\Sigma = \bigcup_{\theta'\in Y'} \sigma(B_{\theta'}) = \bigcup_{x'\in\overline{\Sigma},\theta'\in Y'} \sigma(B_{\theta',x'}).$$

REMARK 7.6. – *The discrete spectrum of $B_{\theta',x'}$ is made of, at most, a countable number of eigenvalues. In some cases it may be reduced to a finite number of eigenvalues, or even be empty, according to the choice of the matrix $A(x,y)$. If $A(x,y)$ depends only on y_N and has a simple form (for example, it takes only a finite number of values), an explicit computation of the discrete spectrum of $B_{\theta',x'}$ can be performed by solving a simple 1-D ordinary differential equation. Such an example shows that the discrete spectrum may be empty or not, depending on the values of $A(x,y)$.*

To prove Theorem 7.2, we recall the following results from [7] concerning two-scale convergence in the sense of boundary layers (for further references on boundary layers in homogenization, *see e.g.* [8], [9], [11], [27], [30]).

PROPOSITION 7.7.

(1) *Let u_ϵ be a sequence in $L^2(\Omega)$ such that*

$$\|u_\epsilon\|_{L^2(\Omega)} \le C\sqrt{\epsilon}.$$

There exists a subsequence, still denoted by u_ϵ, and a limit $u_0(x',y) \in L^2(\Sigma; L^2_\#(KG))$ such that u_ϵ two-scale converges weakly in the sense of boundary layers *to u_0,* i.e.

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega u_\epsilon(x)\varphi(x',\frac{x}{\epsilon})dx = \frac{1}{|KY'|}\int_\Sigma\int_{KG} u_0(x',y)\varphi(x',y)dx'dy$$

for all test functions $\varphi(x',y) \in L^2_\#(G; C(\overline{\Sigma}))$.

(2) *Let u_ϵ be a sequence which two-scale converges weakly in the sense of boundary layers to u_0, and furthermore satisfies*

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{\epsilon}}\|u_\epsilon\|_{L^2(\Omega)} = \frac{1}{|KY'|^{1/2}}\|u_0\|_{L^2(\Sigma\times KG)}.$$

Then, u_ϵ is said to two-scale converge strongly *in the sense of boundary layers to u_0, which means that, for any sequence v_ϵ in $L^2(\Omega)$ which two-scale converges weakly in the sense of boundary layers to a limit $v_0(x',y) \in L^2(\Sigma \times KG)$, one has*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\Omega u_\epsilon(x)v_\epsilon(x)\phi(x',\frac{x}{\epsilon})dx = \frac{1}{|KY'|}\int_\Sigma\int_{KG} u_0(x',y)v_0(x',y)\phi(x',y)dx'dy,$$

for all smooth functions $\phi(x',y) \in C(\overline{\Sigma}; C^c_\#(KG))$.

(3) *Let u_ϵ be a sequence in $H^1_0(\Omega)$ such that*

$$\frac{1}{\sqrt{\epsilon}}\left(\|u_\epsilon\|_{L^2(\Omega)} + \epsilon\|\nabla u_\epsilon\|_{L^2(\Omega)^N}\right) \le C.$$

There exists a subsequence, still denoted by u_ϵ, and a limit $u_0(x',y) \in L^2(\Sigma; H^1_{0\#}(KG))$ such that u_ϵ two-scale converges in the sense of boundary layers to u_0, and $\epsilon\nabla u_\epsilon$ two-scale converges in the sense of boundary layers to $\nabla_y u_0$.

We also need another lemma, very similar to Lemma 4.7, the proof of which is safely left to the reader.

LEMMA 7.8.

(1) *Let $\phi(x', y)$ be a function in $L^2(\Sigma; L^2_\#(KG))$. Then, the sequence $P^K_\epsilon \phi$ two-scale converges strongly in the sense of boundary layers to ϕ.*

(2) *Let $\phi^\epsilon(x', y)$ be a sequence converging weakly to $\phi(x', y)$ in $L^2(\Sigma; L^2_\#(KG))$. Then, the sequence $P^K_\epsilon \phi^\epsilon$ two-scale converges weakly in the sense of boundary layers to ϕ.*

Proof of Theorem 7.2. – Let $\psi_\epsilon(x', y)$ be a sequence converging weakly to $\psi(x', y)$ in $L^2(\Sigma; L^2_\#(KG))$. For any $\phi \in L^2(\Sigma; L^2_\#(KG))$, we need to show that

$$\lim_{\epsilon \to 0} \int_\Sigma \int_{KG} (B^K_\epsilon \phi)\psi_\epsilon dx' dy = \int_\Sigma \int_{KG} (B^K \phi)\psi dx' dy.$$

By definition of B^K_ϵ and since $(E^K_\epsilon)^* = \epsilon^{-1} P^K_\epsilon$, one has

$$\frac{1}{K^{N-1}} \int_\Sigma \int_{KG} (B^K_\epsilon \phi)\psi_\epsilon dx' dy = \epsilon^{-1} \int_\Omega \tilde{S}_\epsilon(P^K_\epsilon \phi)(P^K_\epsilon \psi_\epsilon) dx = \epsilon^{-1} \int_\Omega u_\epsilon (P^K_\epsilon \psi_\epsilon) dx,$$

where $u_\epsilon = \tilde{S}_\epsilon(P^K_\epsilon \phi)$ is the unique solution of

$$\begin{cases} -\epsilon^2 \operatorname{div} A\left(x, \frac{x}{\epsilon}\right) \nabla u_\epsilon + u_\epsilon = P^K_\epsilon \phi & \text{in } \Omega \\ u_\epsilon = 0 & \text{on } \partial\Omega \, . \end{cases}$$

Since $\|P^K_\epsilon \phi\|_{L^2(\Omega)} \leq C\sqrt{\epsilon}$, the following estimate holds

$$\|u_\epsilon\|_{L^2(\Omega)} + \epsilon \|\nabla u_\epsilon\|_{L^2(\Omega)^N} \leq C\sqrt{\epsilon}.$$

Using Proposition 7.7, it is easily seen that u_ϵ two-scale converges weakly in the sense of boundary layers to a limit $u^K(x', y)$ which is the unique solution of (78) in $L^2(\Sigma; H^1_{0\#}(G))$. Furthermore, since $P^K_\epsilon \phi$ two-scale converges strongly to ϕ, a simple computation (similar to that in the proof of Proposition 4.4) shows that u_ϵ two-scale converges strongly too. Finally, using Lemma 7.8, we can pass to the limit

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_\Omega u_\epsilon (P^K_\epsilon \psi_\epsilon) dx = \frac{1}{|KY'|} \int_\Sigma \int_{KG} u^K \psi dx' dy.$$

Defining the operator B^K by $u^K = B^K \phi$, one can easily check its desired properties.

Proof of Proposition 7.5. – The operator $B_{\theta', x'}$ is clearly non-compact since G is unbounded. Let us first characterize its essential spectrum by using the Weyl criterion. Let μ be in $\sigma_{ess}(B_{\theta', x'})$ and v_n be an associated Weyl sequence of eigenvectors, *i.e.*

$$\|v_n\|_{L^2(G)} = 1, \quad v_n \rightharpoonup 0 \text{ weakly in } L^2(G),$$
$$B_{\theta', x'} v_n - \mu v_n = r_n \to 0 \text{ strongly in } L^2(G).$$

Let us check that for any $R > 0$ the sequence v_n converges strongly to 0 in $L^2(G_R)$ with $G_R = G \cap (Y' \times (0, R))$. By definition $w_n = B_{\theta',x'} v_n$ satisfies

$$-\operatorname{div}\left[A\nabla\left(w_n e^{2\pi i\theta'\cdot y'}\right)\right] + w_n e^{2\pi i\theta'\cdot y'} = v_n e^{2\pi i\theta'\cdot y'} \text{ in } G. \tag{82}$$

From equation (82) we deduce that w_n is uniformly bounded in $H^1(G_R)$ and, by Rellich compactness theorem, that w_n is compact in $L^2(G_R)$. Since $v_n = \frac{1}{\mu} w_n + r_n$ converges weakly to 0, we deduce that both v_n and w_n converge strongly to 0 in $L^2(G_R)$. Then, using again equation (82) it is easily seen that w_n converges strongly to 0 in $H^1(G_R)$.

Let $\phi(y_N) \in C^\infty(\mathbb{R})$ be a smooth cut-off function such that $\phi \equiv 0$ on $]-\infty, 1]$ and $\phi \equiv 1$ on $[2; +\infty[$. We build a new Weyl sequence defined by $u_n = \phi v_n$ (remark that $\|\phi v_n\|_{L^2(G)} \to 1$). Defining $t_n = \phi w_n$, it is the solution in the entire band $[0,1]^{N-1} \times]-\infty; +\infty[$ of

$$-\operatorname{div}\left[A\nabla\left(t_n e^{2\pi i\theta'\cdot y'}\right)\right] + t_n e^{2\pi i\theta'\cdot y'} = \phi v_n e^{2\pi i\theta'\cdot y'} + r_n',$$

with

$$r_n' = -w_n \operatorname{div}\left[A\nabla\left(\phi e^{2\pi i\theta'\cdot y'}\right)\right] - 2A\nabla\left(w_n e^{2\pi i\theta'\cdot y'}\right) \cdot \nabla\phi.$$

Since w_n converges strongly to 0 in $H^1_{loc}(G)$ and $\nabla\phi$ has compact support, it implies that r_n' converges strongly to 0 in the band $[0,1]^{N-1} \times]-\infty; +\infty[$. Therefore, u_n is a Weyl sequence for an equation similar to (82), but posed in the whole band $[0,1]^{N-1} \times]-\infty; +\infty[$. It is possible to apply to this equation the Bloch decomposition in the y_N direction, and therefore to prove that its spectrum is nothing but the union of the spectra of the operators $T_{(x',0),(\theta',\theta_N)}$ when θ_N runs in $[0,1]$. This yields that μ belongs to $\bigcup_{0 \le \theta_N \le 1} \sigma(T_{(x',0),(\theta',\theta_N)})$.

To prove the reverse inclusion, $\sigma(T_{(x',0)(\theta',\theta_N)}) \subset \sigma(B_{\theta',x'})$, we take an eigenvalue μ and eigenvector u of $T_{(x',0)(\theta',\theta_N)}$, normalized by $\|u\|_{L^2(Y)} = 1$. We build a Weyl sequence v_n for $B_{\theta',x'}$ defined by

$$v_n(y) = \frac{u(y)\psi_n(y_N)}{\|u\psi_n\|_{L^2(G)}},$$

where ψ_n is a sequence of smooth cut-off functions given by

$$\begin{cases} \psi_n(y_N) = y_N & \text{on } 0 \le y_N \le 1, \\ \psi_n(y_N) = 1 & \text{on } 1 \le y_N \le n, \\ \psi_n(y_N) = n + 1 - y_N & \text{on } n \le y_N \le n+1, \\ \psi_n(y_N) = 0 & \text{on } n + 1 \le y_N. \end{cases}$$

Remark that $\|u\psi_n\|_{L^2(G)}$ goes to $+\infty$ with n. It is easily seen that v_n converges weakly to 0 in $L^2(G)$, while the function r_n defined by

$$-\operatorname{div}\left[A\nabla\left(v_n e^{2\pi i\theta'\cdot y'}\right)\right] + v_n e^{2\pi i\theta'\cdot y'} = \frac{1}{\mu} v_n e^{2\pi i\theta'\cdot y'} + r_n$$

converges strongly to 0 in $L^2(G)$. Therefore, μ belongs to the essential spectrum of $B_{\theta',x'}$.

We now prove that the eigenvectors associated to eigenvalues in $\sigma_{disc}(B_{\theta',x'})$ decay exponentially in G. Our argument is by contradiction of the Weyl criterion. Let $\mu \in \sigma_{disc}(B_{\theta',x'})$ and $u(y)$ be an associated normalized eigenvector. Let ϕ_n be a sequence of smooth cut-off functions defined by

$$\begin{cases} \phi_n(y_N) = 0 & \text{on } 0 \leq y_N \leq n, \\ \phi_n(y_N) = y_N - n & \text{on } n \leq y_N \leq n+1, \\ \phi_n(y_N) = 1 & \text{on } n+1 \leq y_N. \end{cases}$$

Let $u_n = \frac{u\phi_n}{\|u\phi_n\|_{L^2(G)}}$. Clearly u_n converges weakly to 0 in $L^2(G)$. However, any subsequence of u_n can not be a Weyl sequence for μ, since μ belongs to $\sigma_{disc}(B_{\theta',x'})$ which is disconnected from $\sigma_{ess}(B_{\theta',x'})$. Defining $r_n = B_{\theta',x'}u_n - \mu u_n$, this implies that there exists a positive constant C and an integer n_0 such that

$$\forall n \geq n_0 \quad \|r_n\|_{L^2(G)} \geq C.$$

Since u is an eigenvector, an easy computation yields

$$\begin{aligned} -\operatorname{div}\Big[A\nabla\Big(r_n e^{2\pi i\theta'\cdot y'}\Big)\Big] + r_n e^{2\pi i\theta'\cdot y'} = & \frac{\mu}{\|u\phi_n\|_{L^2(G)}}\Big(u \operatorname{div}\Big[A\nabla\Big(\phi_n e^{2\pi i\theta'\cdot y'}\Big)\Big] \\ & +2A\nabla u \cdot \nabla\Big(\phi_n e^{2\pi i\theta'\cdot y'}\Big)\Big). \end{aligned}$$

Multiplying this equation by r_n and having in mind that $\nabla\phi_n$ has compact support in $G_n = G \cap (Y' \times (n, n+1))$ leads to the estimate

$$\|r_n\|_{L^2(G)} + \|\nabla r_n\|_{L^2(G)^N} \leq \frac{C}{\|u\phi_n\|_{L^2(G)}}\big(\|\nabla u\|_{L^2(G_n)^N} + \|u\|_{L^2(G_n)}\big).$$

Since $\|r_n\|_{L^2(G)} \geq C$, we deduce

$$\|u\phi_n\|_{L^2(G)} \leq C\big(\|\nabla u\|_{L^2(G_n)^N} + \|u\|_{L^2(G_n)}\big).$$

Since u satisfies the spectral equation in G it is not difficult to check that

$$\|\nabla u\|_{L^2(G_n)^N} \leq C\|u\|_{L^2(G_{n-1}\cup G_n \cup G_{n+1})}.$$

Therefore, defining a function $F(n) = \|u\|^2_{L^2(Y'\times(n,+\infty))}$, we have proved

$$F(n+1) \leq C(F(n-1) - F(n+2)).$$

Since F is decreasing, this implies $F(n+3p) \leq \left(\frac{C}{C+1}\right)^p F(n)$ which yields the exponential decay of u when y_N goes to $+\infty$.

It remains to prove that the eigenvalues in $\sigma_{disc}(B_{\theta',x'})$ are locally continuous with respect to (x', θ'). Labeling these discrete eigenvalues by decreasing order, this is a result of a classical spectral perturbation theorem (*see* Theorem 7.9 below, the proof of which may be found in *e.g.* [26], [28], [43]). The continuity of the eigenvalues of $\sigma_{disc}(B_{\theta',x'})$

is only local since the labeling of the eigenvalues allows for jumps when one discrete eigenvalue happens to merge into the essential spectrum as θ' varies. However, because $\sigma_{ess}(B_{\theta',x'}) = \bigcup_{0\leq\theta_N\leq 1} \sigma(T_{(\theta',\theta_N),(x',0)})$, the essential spectrum of $B_{\theta',x'}$ is continuous (considered as a subset of $\mathbb{R}^+$), and so is its entire spectrum.

THEOREM 7.9. – *Let A_n be a sequence of bounded operators in a Hilbert separable space which converges uniformly to a limit operator A in $\mathcal{L}(H)$. Let Γ be a smooth compact curve in the complex plane which encloses a finite number of eigenvalues in $\sigma_{disc}(A)$ and does not intersect $\sigma(A)$. There exists an integer n_0 such that for any $n \geq n_0$, the same curve Γ encloses the same number of eigenvalues (including multiplicities) in $\sigma_{disc}(A_n)$ and does not intersect $\sigma(A_n)$.*

7.2. Completeness of the boundary layer spectrum

In this subsection we assume that Ω is a rectangle with integer dimensions, *i.e.*

$$\Omega = \prod_{i=1}^{N}]0; L_i[\,, \quad \text{and} \quad L_i \in \mathbb{N}^*. \tag{83}$$

The sequence of small parameters ϵ is restricted to be of the type

$$\epsilon_n = \frac{1}{n},\ n \in \mathbb{N}^*, \tag{84}$$

in such a way that Ω is the union of a finite number of *entire periodic cells* $Y_i^{\epsilon_n}$. To simplify the notations, we shall not indicate the dependence on n and simply denotes by ϵ the particular sequence defined in (84).

REMARK 7.10. – *Remark that the assumption on the geometry of Ω can be slightly relaxed. Any polygonal domain with faces parallel to the axis (*i.e.* the normal is everywhere one of the basis vectors) and having vertex with integer coordinates could equally be considered. The crucial point is that there still exists some periodicity of semi-infinite bands normal to the boundary. The general case of a non-polygonal domain and of any possible sequence ϵ is not treated here and is a difficult open question.*

In the previous subsection, we proved that

$$\sigma_\Sigma \subset \lim_{\epsilon\to 0} \sigma(\tilde{S}_\epsilon),$$

where σ_Σ is the boundary layer spectrum associated to the surface Σ, defined by (81) and $\tilde{S}_\epsilon$ is the operator defined by (72). Remark that, *due to our hypotheses on the domain Ω and on the sequence ϵ*, the surface Σ can be any of the faces of Ω defined by

$$\prod_{\substack{j=1\\ j\neq i}}^{N}]0; L_j[\times\{0\} \quad \text{or} \quad \prod_{\substack{j=1\\ j\neq i}}^{N}]0; L_j[\times\{L_i\} \quad \text{for } 1 \leq i \leq N.$$

Of course, the analysis of the previous subsection can be repeated for any other lower dimensional manifolds (edges, corners, etc.) which compose the boundary of Ω. For $0 \leq m \leq N-1$, let us define the m-dimensional parts of $\partial\Omega$

$$\Sigma_{m,\tau} = \prod_{j=1}^{m}]0; L_{\tau(j)}[\times \prod_{j=m+1}^{N} \{x_{\tau(j)} = 0 \text{ or } L_{\tau(j)}\},$$

where τ is any permutation of $\{1, 2, ..., N\}$. There are $2^{N-m} C_N^{N-m}$ m-dimensional manifolds of the type $\Sigma_{m,\tau}$. A simple adaptation of the two-scale convergence in the sense of boundary layers for such manifolds allows to prove that, for any m and τ,

$$\sigma_{\Sigma_{m,\tau}} \subset \lim_{\epsilon \to 0} \sigma(\tilde{S}_\epsilon),$$

where $\sigma_{\Sigma_{m,\tau}}$ is the spectrum of a family of limit problems posed, not in a semi-infinite band G, but rather in a periodic domain bounded in the variables $x_{\tau(1)}, ..., x_{\tau(m)}$ and unbounded with respect to the other variables. Eventually, defining the union of all these spectra

$$\sigma_{\partial\Omega} = \bigcup_{m,\tau} \sigma_{\Sigma_{m,\tau}}, \tag{85}$$

we deduce from the geometric assumptions on Ω and ϵ that

$$\sigma_{\partial\Omega} \subset \lim_{\epsilon \to 0} \sigma(\tilde{S}_\epsilon). \tag{86}$$

Comparing our results (17) and (86), a completeness result amounts to link the two definitions of the boundary layer spectra $\sigma_{\partial\Omega}$ and σ_{boundary}.

THEOREM 7.11. – *For the sequence ϵ_n defined by* (84), the boundary layer spectrum satisfies

$$\sigma_{\text{boundary}} \subset \sigma_{\partial\Omega}.$$

Therefore, the limit spectrum of the sequence $\tilde{S}_{\epsilon_n}$ is precisely made of two parts, the Bloch and the boundary layer spectrum

$$\lim_{\epsilon_n \to 0} \sigma(\tilde{S}_{\epsilon_n}) = \sigma_{\text{Bloch}} \cup \sigma_{\partial\Omega},$$

where the boundary layer spectrum $\sigma_{\partial\Omega}$ is explicitly defined by (85).

REMARK 7.12. – *Theorem* 7.11 *does not state that σ_{boundary}, defined by* (16), *and $\sigma_{\partial\Omega}$ coincide. Indeed, we have shown in Proposition* 7.5 *that $\sigma_{\partial\Omega}$ contains some parts of the Bloch spectrum. It is not clear whether σ_{boundary} contains these parts of the Bloch spectrum too. The comparison of $\sigma_{\partial\Omega}$ and σ_{boundary} is definitely a very difficult question. Note, however, that $\sigma_{\partial\Omega}$ may contain eigenvalues which do not belong to σ_{Bloch}, according to Remark* 7.6.

To prove this completeness result, we need an intermediate result in the spirit of Section 5.

THEOREM 7.13. – *Let us consider* Ω *as a cylindrical domain defined by* $\Omega = \Sigma\times]0;L[$, *with* Σ *a bounded open set in* $\mathbb{R}^{N-1}$ *and* $L > 0$. *Consider a sequence of eigenvalues* μ_ϵ *and eigenvectors* $v_\epsilon \in H_0^1(\Omega)$ *such that*

$$(87) \qquad \|v_\epsilon\|_{L^2(\Omega)} = 1, \quad \lim_{\epsilon\to 0} \mu_\epsilon = \mu,$$

$$(88) \qquad \begin{cases} -\epsilon^2 \operatorname{div}\left[A(x, \frac{x}{\epsilon})\nabla v_\epsilon\right] + v_\epsilon = \frac{1}{\mu_\epsilon} v_\epsilon & \text{in } \Omega, \\ v_\epsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that μ *belongs to* σ_{boundary}, i.e. *for any* $n \geq 0$, *there exists a constant* $C(n)$ *such that*

$$(89) \qquad \|v_\epsilon d(x,\partial\Omega)^n\|_{L^2(\Omega)} + \epsilon\|(\nabla v_\epsilon)d(x,\partial\Omega)^n\|_{L^2(\Omega)^N} \leq C(n)\epsilon^n.$$

Assume further that there exists a $(N-1)$*-dimensional open set* σ, *with* $\overline{\sigma} \subset \Sigma$, *a positive number* l, *with* $0 < l < L$, *and a positive constant* c *such that*

$$(90) \qquad \lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\sigma\times]0,l[)} \geq c > 0.$$

Then μ *belongs to the boundary layer spectrum associated to the surface* Σ

$$\mu \in \sigma_\Sigma,$$

where σ_Σ *is defined by* (81).

Let us admit for a moment Theorem 7.13, as well as its generalizations concerning all other manifolds $\Sigma_{m,\tau}$ making up the boundary $\partial\Omega$. We are in a position to complete the

Proof of Theorem 7.11. – Let $\mu \in \sigma_{\text{boundary}}$. By definition there exists a subsequence (still denoted by ϵ) of eigenvalues μ_ϵ and eigenvectors v_ϵ of $\tilde{S}_\epsilon$ such that

$$\tilde{S}_\epsilon v_\epsilon = \mu_\epsilon v_\epsilon \quad \text{with} \quad \|v_\epsilon\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \lim_{\epsilon\to 0} \mu_\epsilon = \mu,$$

and, for all subset ω satisfying $\overline{\omega} \subset \Omega$,

$$\lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\omega)} = 0.$$

If there exists a $(N-1)$-dimensional open subset σ_i, compactly embedded in $\prod_{j=1, j\neq i}^{N}]0; L_j[$, a positive length $0 < l_i < L_i$, a positive constant $c > 0$, and another subsequence (still denoted by ϵ) such that

$$(91) \qquad \lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\sigma_i\times]0,l_i[)} \geq c \quad \text{or} \quad \lim_{\epsilon\to 0} \|v_\epsilon\|_{L^2(\sigma_i\times]l_i,L_i[)} \geq c,$$

then, by application of Theorem 7.13, the limit eigenvalue belongs to $\sigma_{\partial\Omega}$ as desired.

If (91) does not hold true for any such σ_i, l_i, c and subsequence ϵ, it implies that the L^2-norm of v_ϵ concentrates near the lower-dimensional edges of the rectangle Ω. In this case, we repeat the above argument with a $(N-2)$-dimensional open set included in one of the set $\Sigma_{N-2,\tau}$, and so on up to the 0-dimensional set made of one of the vertex of Ω. A tedious but simple induction argument on the dimension m shows that there exists at least a dimension $0 \leq m \leq N-1$, a permutation τ, positive lengths $(l_{\tau(j)})_{m+1 \leq j \leq N}$, a positive constant c, and a subsequence ϵ such that

$$\lim_{\epsilon \to 0} \|v_\epsilon\|_{L^2(\omega)} \geq c > 0,$$

with $\omega \subset \Omega$ of the type

$$\omega = \sigma \times \prod_{j=m+1}^{N} \left( ]0, l_{\tau(j)}[\text{ or }]l_{\tau(j)}, L_{\tau(j)}[\right) \quad \text{and} \quad \overline{\sigma} \subset \prod_{j=1}^{m}]0; L_{\tau(j)}[.$$

Then, applying an adequate generalization of Theorem 7.13, this proves that the limit eigenvalue belongs to $\sigma_{\partial\Omega}$.

Proof of Theorem 7.13. – Let ψ_N and ψ' be two smooth cut-off functions satisfying

$$\begin{cases} \psi_N(x_N) \geq 0 & \forall x_N \in \mathbb{R}, \\ \psi_N(x_N) \equiv 1 & x_N < L/3, \\ \psi_N(x_N) \equiv 0 & x_N > 2L/3, \end{cases}$$

and

$$\begin{cases} \psi'(x') \geq 0 & \text{in} \quad \mathbb{R}^{N-1}, \\ \psi'(x') \equiv 1 & \text{in} \quad \sigma, \\ \psi'(x') \equiv 0 & \text{outside} \quad \Sigma. \end{cases}$$

Let us define a sequence u_ϵ, supported away from all boundaries of Ω except Σ, by

$$u_\epsilon = \frac{\psi(x_N)\psi'(x')v_\epsilon(x)}{\|\psi(x_N)\psi'(x')v_\epsilon(x)\|_{L^2(\Omega)}}. \tag{92}$$

It is not difficult to check that the sequence u_ϵ is a sequence of quasi eigenvectors in $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times \mathbb{R}^+$ in the sense that it satisfies

$$\begin{cases} -\epsilon^2 \operatorname{div}\left[A\left(x, \frac{x}{\epsilon}\right)\nabla u_\epsilon\right] + u_\epsilon = \frac{1}{\mu_\epsilon} u_\epsilon + r_\epsilon & \text{in } \mathbb{R}^N_+, \\ u_\epsilon = 0 & \text{on } x_N = 0, \end{cases} \tag{93}$$

where r_ϵ is a remainder term which satisfies

$$\lim_{\epsilon \to 0} \frac{\langle r_\epsilon, w_\epsilon \rangle_{H^{-1}, H^1_0(\mathbb{R}^N_+)}}{\|w_\epsilon\|_{L^2(\mathbb{R}^N_+)} + \epsilon \|\nabla w_\epsilon\|_{L^2(\mathbb{R}^N_+)^N}} = 0,$$

for all non-zero sequences $w_\epsilon \in H^1_0(\mathbb{R}^N_+)$.

Remark that this definition of quasi eigenvectors in $\mathbb{R}^N_+$ is slightly different from that in $\mathbb{R}^N$ (*cf.* Definition 5.4) since it features a Dirichlet boundary condition on $x_N = 0$. Let β_ϵ be a sequence of intermediate scales such that $\epsilon \ll \beta_\epsilon \ll 1$ and β_ϵ is an entire multiple of ϵ, *i.e.*

$$\lim_{\epsilon\to 0} \beta_\epsilon = 0, \quad \lim_{\epsilon\to 0} \frac{\epsilon}{\beta_\epsilon} = 0, \quad \text{and} \quad \frac{\beta_\epsilon}{\epsilon} = p_\epsilon \in \mathbb{N}.$$

The domain Ω is covered by a mesh of non-overlapping cubes $(P_i^\epsilon)_{1\le i\le n(\beta_\epsilon)}$ of the type $[0,\beta_\epsilon]^N$. The number of such cubes is $n(\beta_\epsilon)$, which is of the order of $\frac{|\Omega|}{\beta_\epsilon^N}$. We denote by x_i^ϵ the center of each cube P_i^ϵ, and by $i(\epsilon)$ the index such that the L^2-norm of u_ϵ is maximum on the cube $P_{i(\epsilon)}^\epsilon$

$$\|u_\epsilon\|_{L^2(P_{i(\epsilon)}^\epsilon)} = \max_{1\le i\le n(\beta_\epsilon)} \|u_\epsilon\|_{L^2(P_i^\epsilon)}. \tag{94}$$

Since $\sum_{1\le i\le n(\beta_\epsilon)} \|u_\epsilon\|^2_{L^2(P_i^\epsilon)} = 1$, we deduce that there exists a positive constant $C > 0$ such that

$$\|u_\epsilon\|_{L^2(P_{i(\epsilon)}^\epsilon)} \ge C\beta_\epsilon^{N/2}. \tag{95}$$

Since $x_{i(\epsilon)}^\epsilon$ runs in the compact set $\overline{\Omega}$, there exists a subsequence, still denoted by ϵ, and a limit point $x_0 \in \overline{\Omega}$, such that $x_{i(\epsilon)}^\epsilon$ converges to x_0. Moreover, due to the estimate (89), x_0 must belong to $\sigma \subset \Sigma$. If it were not true, one would obtain a contradiction in (89) for $n \ge N/2$.

In order to localize u_ϵ around x_0, we define a smooth function $\phi \in \mathcal{D}(\mathbb{R}^N)$ such that

$$\begin{cases} \phi \ge 0 & \text{in} \quad \mathbb{R}^N, \\ \phi \equiv 1 & \text{in} \quad [-1/2,+1/2]^N, \\ \phi \equiv 0 & \text{outside} \quad [-1;+1]^N. \end{cases}$$

Introducing the cut-off function

$$\phi_\epsilon(x) = \phi\left(\frac{x - x_{i(\epsilon)}^\epsilon}{\beta_\epsilon}\right),$$

we define a function $\tilde{u}_\epsilon$ in $H_0^1(\mathbb{R}^N_+)$ by

$$\tilde{u}_\epsilon = \frac{\phi_\epsilon u_\epsilon}{\|\phi_\epsilon u_\epsilon\|_{L^2(\mathbb{R}^N)}}. \tag{96}$$

As in the proof of Proposition 5.9, an adequate choice of the intermediate scale β_ϵ allows to prove that $\tilde{u}_\epsilon$ is also a sequence of quasi eigenvectors in $\mathbb{R}^N_+$ for the matrix $A(x_0, \frac{x}{\epsilon})$, *i.e.* it satisfies

$$\begin{cases} -\epsilon^2 \operatorname{div}\left[A(x_0, \frac{x}{\epsilon})\nabla \tilde{u}_\epsilon\right] + \tilde{u}_\epsilon = \frac{1}{\mu_\epsilon}\tilde{u}_\epsilon + r_\epsilon^0 & \text{in } \mathbb{R}^N_+, \\ \tilde{u}_\epsilon = 0 & \text{on } x_N = 0, \end{cases} \tag{97}$$

where r_ϵ^0 is a remainder term which satisfies

$$\lim_{\epsilon\to 0} \frac{\langle r_\epsilon^0, w_\epsilon\rangle_{H^{-1},H_0^1(\mathbb{R}_+^N)}}{\|w_\epsilon\|_{L^2(\mathbb{R}_+^N)} + \epsilon\|\nabla w_\epsilon\|_{L^2(\mathbb{R}_+^N)^N}} = 0,$$

for all non-zero sequences $w_\epsilon \in H_0^1(\mathbb{R}_+^N)$.

At this point we could apply the spectral and Bloch decompositions to the sequence $\tilde{u}_\epsilon$ (specialized for the case of the half-space domain $\mathbb{R}_+^N$), and mimic the proof of Theorem 5.2. However, we would run into serious troubles since the spectral decomposition is not discrete: the operators $B_{\theta',x_0'}$ have both discrete and essential spectrum. Therefore, one must use integration with respect to the spectral family rather than summation over discrete frequencies. This leads to intricate problems when some eigenvalues change type (discrete or essential) as θ' varies. To avoid these difficulties, we use a different strategy based on a rescaling or blow-up argument.

By the change of variables $y = x/\epsilon$ we define

$$U_\epsilon(y) = \epsilon^{N/2}\tilde{u}_\epsilon(\epsilon y)$$

which is easily seen to be normalized, $\|U_\epsilon\|_{L^2(\mathbb{R}_+^N)} = 1$, and solution of

$$\text{(98)} \qquad \begin{cases} -\operatorname{div}_y[A(x_0,y)\nabla_y U_\epsilon] + U_\epsilon = \frac{1}{\mu_\epsilon}U_\epsilon + R_\epsilon & \text{in } \mathbb{R}_+^N, \\ U_\epsilon = 0 & \text{on } y_N = 0, \end{cases}$$

where R_ϵ is a remainder term which converges strongly to 0 in $H^{-1}(\mathbb{R}_+^N)$. Since U_ϵ is bounded in $H_0^1(\mathbb{R}_+^N)$, there exists a limit function U such that, up to a subsequence, U_ϵ converges weakly to U. Multiplying equation (98) by a test function and passing to the limit yields that U is a solution of

$$\text{(99)} \qquad \begin{cases} -\operatorname{div}_y[A(x_0,y)\nabla_y U] + U = \frac{1}{\mu}U & \text{in } \mathbb{R}_+^N, \\ U = 0 & \text{on } y_N = 0. \end{cases}$$

Let us introduce an operator B^∞ defined by

$$\begin{aligned} B^\infty : L^2(\mathbb{R}_+^N) &\to L^2(\mathbb{R}_+^N) \\ F(y) &\to V(y) \end{aligned}$$

where $V(y)$ is the unique solution in $H_0^1(\mathbb{R}_+^N)$ of

$$\begin{cases} -\operatorname{div}_y[A(x_0,y)\nabla_y V(y)] + V(y) = F(y) & \text{in } \mathbb{R}_+^N \\ V(y) = 0 & \text{on } y_N = 0. \end{cases}$$

Loosely speaking B^∞ is the limit, as K goes to infinity, of B^K defined in Theorem 7.2. It is easily checked that $\delta_\epsilon = B^\infty U_\epsilon - \mu U_\epsilon$ converges strongly to 0 in $H_0^1(\mathbb{R}_+^N)$. If the solution U of (99) satisfies $U \neq 0$, we have found an eigenvector of B^∞ for the eigenvalue μ. On the other hand, if $U = 0$, this implies that U_ϵ is a Weyl sequence of B^∞ for the eigenvalue μ which therefore belongs to its essential spectrum. In both cases, a simple Bloch decomposition of (99) shows that μ must belong to the spectrum σ_Σ.

REMARK 7.14. – *Let us remark that Theorem* 7.13 *is valid for any choice of the sequence* ϵ *and not only for the particular sequence* ϵ_n *defined in* (84). *The interested reader will not fail to notice that the present proof of the completeness result is different from that of section* 5 *where we used the concept of Bloch measures in order to prove a similar completeness result by means of an energetic method. Here, we propose a new proof, based on a rescaling or blow-up argument, which is simpler, although less precise, and which could equally be applied in section* 5. *We use this new argument (already introduced in our work* [7]) *because the spectral decomposition of* v_ϵ *and the global continuity of the discrete eigenvalues (with respect to the Bloch parameter* θ'*) are not obvious.*

7.3. Analysis of the corner spectrum in 2-D

In the previous subsection the boundary layer spectrum $\sigma_{\partial\Omega}$ was defined as the union of all spectra of the type σ_Σ where Σ is any lower dimensional manifold composing the boundary $\partial\Omega$. When Σ is a $(N-1)$-dimensional hyperplane, a complete characterization of σ_Σ has been given in details. However, for lower dimensional manifolds we have not been very precise in the generalization of the limit spectrum σ_Σ to the case of edges, corners, and so on. The purpose of this subsection is to give a brief account of this generalization when analyzing the *corner spectrum* in two dimensions. Restricting ourselves to plane square domain Ω has the advantage of simplifying the exposition without loosing much generality.

The domain Ω is from now on a rectangle with integer dimensions, *i.e.*

$$\Omega =]0; L_1[\times]0; L_2[.$$

We describe the limit spectrum associated to the corner located at the origin. We define the upper right quarter of space $Q^+ = \mathbb{R}^+ \times \mathbb{R}^+$. We extend the operator $\tilde{S}_\epsilon$, defined by (72), to the space $L^2(Q^+)$. Remark that it is a space of "corner boundary layers" without any periodic oscillations. The extented operator $C_\epsilon \in \mathcal{L}(L^2(Q^+))$ is defined by

$$C_\epsilon = E_\epsilon \tilde{S}_\epsilon P_\epsilon, \tag{100}$$

where P_ϵ is a projection from $L^2(Q^+)$ onto $L^2(\Omega)$, defined, for any $\phi(y) \in L^2(Q^+)$, by $(P_\epsilon\phi)(x) = \epsilon^{-2}\phi(\frac{x}{\epsilon})$ restricted to Ω, and E_ϵ is an extension from $L^2(\Omega)$ into $L^2(Q^+)$, defined, for any $f(x) \in L^2(\Omega)$, by $(E_\epsilon f)(y) = \epsilon^2 f(\epsilon y)$ extended by 0 in $Q^+ \setminus \epsilon^{-1}\Omega$. One can easily check that P_ϵ and E_ϵ are adjoint one from another and satisfy

$$P_\epsilon E_\epsilon = \mathrm{Id}_{L^2(\Omega)}, \quad (P_\epsilon)^* = E_\epsilon.$$

Furthermore, the sequence $E_\epsilon P_\epsilon$ converges strongly to the identity in $\mathcal{L}(L^2(Q^+))$.

THEOREM 7.15. – *The sequence* C_ϵ *converges strongly in* $\mathcal{L}(L^2(Q^+))$ *to a self-adjoint and non-compact operator* C *defined, for any* $\phi \in L^2(Q^+)$, *by* $C\phi = u$ *the unique solution in* $H^1(Q^+)$ *of*

$$\begin{cases} -\mathrm{div}_y[A(0,y)\nabla_y u(y)] + u(y) = \phi(y) & \text{in } Q^+ \\ u(y) = 0 & \text{on } y_1 = 0 \text{ and } y_2 = 0. \end{cases} \tag{101}$$

The spectrum of C satisfies

$$\sigma(C) \subset \lim_{\epsilon \to 0} \sigma(\tilde{S}_\epsilon).$$

Problem (101) can not be simplified by using Bloch waves since Q^+ is not a periodic domain. The operator C is non-compact since the quarter space Q^+ is unbounded. Therefore its spectrum can be decomposed into its discrete and essential parts

$$\sigma(C) = \sigma_{disc}(C) \cup \sigma_{ess}(C).$$

PROPOSITION 7.16. – *The essential spectrum of C is given by*

$$\sigma_{ess}(C) = \bigcup_{0 \le \theta_1 \le 1,\ 0 \le \theta_2 \le 1} \sigma(T_{(\theta_1,\theta_2),0}) \bigcup_{0 \le \theta_1 \le 1} \sigma(B_{\theta_1,0}) \bigcup_{0 \le \theta_2 \le 1} \sigma(B_{\theta_2,0}),$$

where $T_{\theta,x}$ is the operator defined in Proposition 4.12, $B_{\theta_1,x}$ and $B_{\theta_2,x}$ are the operators defined in Proposition 7.5 respectively for the boundaries $y_2 = 0$ and $y_1 = 0$. Furthermore, any eigenvector for a discrete eigenvalue in $\sigma_{disc}(C)$ is exponentially decreasing when y_1 or y_2 goes to infinity.

Remark that the essential spectrum of C has again a band structure. However, there may be discrete eigenvalues too in $\sigma(C)$. Therefore, the boundary layer spectrum $\sigma_{\partial\Omega}$ defined by (85) may contain isolated discrete eigenvalues.

The proofs of Theorem 7.15 and Proposition 7.16 are very similar to that of Theorem 7.2 and Proposition 7.5 and are left to the reader. Remark that the matrix $A(x, y)$ and the operator $T_{\theta,x}$ are both evaluated at the origin $x = 0$ in the above results. Of course, there may be eigenvalues in the discrete spectrum of C which are not in any of the previous limit spectra.

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