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Homogenization and localization for a 1-D eigenvalue problem in a periodic medium with an interface

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Abstract. In one space dimension we address the homogenization of the spectral problem for a singularly perturbed diffusion equation in a periodic medium. Denoting by ϵ the period, the diffusion coefficient is scaled as ϵ^2. The domain is made of two purely periodic media separated by an interface. Depending on the connection between the two cell spectral equations, three different situations arise when ϵ goes to zero. First, there is a global homogenized problem as in the case without an interface. Second, the limit is made of two homogenized problems with a Dirichlet boundary condition on the interface. Third, there is an exponential localization near the interface of the first eigenfunction.

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1. Introduction

This paper is devoted to the homogenization of the eigenvalue problem for a singularly perturbed diffusion equation in a periodic medium. Although this problem is of interest in higher-space dimensions, we restrict ourselves to the one-dimensional case because of the difficulty of the analysis. In particular, one of our key tools is the theory of Hill's ordinary differential equation [14] for which there is no equivalent in higher dimensions. Denoting the period by ϵ, the diffusion coefficient is assumed to be of the order of ϵ^2. Thus, we consider the following model:

$$\begin{cases} -\epsilon^2 \dfrac{d}{dx}\left(a\left(x,\dfrac{x}{\epsilon}\right)\dfrac{d}{dx}\phi^\epsilon\right) + \Sigma\left(x,\dfrac{x}{\epsilon}\right)\phi^\epsilon = \lambda^\epsilon \sigma\left(x,\dfrac{x}{\epsilon}\right)\phi^\epsilon \text{ in } \Omega, \\ \phi^\epsilon = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ^ϵ, ϕ^ϵ is an eigenvalue and eigenfunction (throughout this paper, the eigenfunctions are normalized by $\|\phi^\epsilon\|_{L^2(\Omega)} = 1$). In (1.1) the coefficients are periodic of period 1 with respect to the fast variable x/ϵ. The general study of the homogenization of (1.1) is far from being complete. When the coefficients are not rapidly oscillating (i.e., they depend on the slow variable x but not on x/ϵ), it is a problem of singular perturbation (without homogenization) which is quite well understood

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now in any space dimension (see, e.g., [23]). When the coefficients are purely periodic functions (i.e., they depend solely on x/ϵ), the homogenization of (1.1) (and similar models in higher dimension) has been achieved in [2], [3], [5]. In the case of smooth coefficients with a concentration hypothesis, partial results have recently been obtained in [6] (again in any space dimension). Here we focus on the different case (of practical as well as theoretical importance) where the coefficients are discontinuous. More precisely, we focus on the simplest possible model in this context, assuming that the domain is composed of two periodical media separated by an interface.

The domain Ω is of the form $(-l, L)$, where l and L are strictly positive constants, and we introduce the two sub-domains $\Omega_1 = (-l, 0)$ and $\Omega_2 = (0, L)$ separated by an interface located at the point 0. Denoting by $\chi_i(x)$ the characteristic function of Ω_i (satisfying $\chi_1 + \chi_2 = 1$ and $\chi_1\chi_2 = 0$ in Ω), the coefficients are assumed to be given as:

$$\begin{cases} a(x, y) = \chi_1(x)a_1(y) + \chi_2(x)a_2(y), \\ \Sigma(x, y) = \chi_1(x)\Sigma_1(y) + \chi_2(x)\Sigma_2(y), \\ \sigma(x, y) = \chi_1(x)\sigma_1(y) + \chi_2(x)\sigma_2(y). \end{cases} \tag{1.2}$$

All functions $a_1, a_2, \Sigma_1, \Sigma_2, \sigma_1$ and σ_2 are assumed to be measurable, 1-periodic, bounded from above and below by positive constants. Under these assumptions, it is well known that Equation (1.1) admits a countable infinite number of non-trivial solutions $(\lambda_m^\epsilon, \phi_m^\epsilon)_{m\geq 1}$. By standard regularity results, each eigenfunction ϕ_m^ϵ belongs to $H_0^1(\Omega)\cap C^{0,s}(\Omega)$, with $s > 0$, and by the Krein–Rutman theorem the first eigenvalue is simple and the corresponding eigenfunction can be chosen positive. Because of this property, the first eigenpair has a special physical signification, and we are mostly interested in its behavior, although the case of higher-level eigenpairs is also treated in some occasions.

The motivation for studying this model comes from several applications. First, it can be seen as a semi-classical limit problem for a Schrödinger-type equation with periodic potential, as well as a periodic metric (this is the so-called ground-state asymptotic problem, see, e.g., [17], [23]). Second, it plays an important role in the uniform controllability of the wave equation (see, e.g., [11]). Third, and this is our main motivation, it is a simple model for computing the power distribution in a nuclear reactor core. This is the so-called criticality problem for the one-group neutron diffusion equation (for more details, we refer to [3], [9] and references therein). In all these applications, the assumption of a purely periodic medium (i.e., no dependence on x of the coefficients) is much too strong. On the other hand the coefficients are not smoothly varying but exhibit jumps at material interfaces. This makes Model (1.1) with Assumptions (1.2) physically relevant.

The limit behavior of (1.1) is mainly governed by the first eigenpair (ψ_i, μ_i) in the unit cell of Ω_i, $i = 1, 2$, solution of

$$\begin{cases} -\dfrac{d}{dy}\left(a_i(y)\dfrac{d}{dy}\psi_i\right) + \Sigma_i(y)\psi_i = \mu_i\sigma_i(y)\psi_i \text{ in } [0, 1], \\ y \to \psi_i(y) \quad \text{1-periodic and positive.} \end{cases} \tag{1.3}$$

Before we explain our main results, let us recall what was already proved in [5] in the purely periodic case, namely when $a_1 = a_2$, $\Sigma_1 = \Sigma_2$, and $\sigma_1 = \sigma_2$. Asymptotically, the macroscopic trend of ϕ^ϵ is given by an homogeneous eigenvalue problem, whereas its oscillatory behavior is governed by $\psi_1(\frac{x}{\epsilon})$ (we call this a factorization principle). More precisely, the result of [5] is:

Theorem 1.1. *Assuming that $a_2 = a_1$, $\Sigma_2 = \Sigma_1$, and $\sigma_2 = \sigma_1$, the m^{th} eigenpair $\lambda_m^\epsilon, \phi_m^\epsilon$ of (1.1) satisfies*

$$\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi_1\left(\frac{x}{\epsilon}\right) \text{ and } \lambda_m^\epsilon = \mu_1 + \epsilon^2 \nu_m + o(\epsilon^2),$$

where, up to a sub-sequence, the sequence u_m^ϵ converges weakly in $H_0^1(\Omega)$ to u_m, and (ν_m, u_m) is the m^{th} eigenvalue and eigenvector for the homogenized problem

$$\begin{cases} -\overline{D}\dfrac{d^2}{dx^2}u_m = \nu_m\,\overline{\sigma}u_m & \text{in } \Omega, \\ u_m = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The homogenized coefficients are given by

$$\overline{D} = \int_0^1 a_1(y)\psi_1^2(y)\left(1 + \frac{d\xi}{dy}(y)\right)dy \text{ and } \overline{\sigma} = \int_0^1 \sigma_1(y)\psi_1^2(y)dy, \tag{1.5}$$

where the function ξ is the solution of

$$\begin{cases} -\dfrac{d}{dy}\left(a_1(y)\psi_1^2(y)\left(\dfrac{d\xi}{dy}+1\right)\right) = 0 & \text{in } [0,1], \\ pny \to \xi(y) \quad 1\text{-periodic}. \end{cases} \tag{1.6}$$

Let us summarize our results in the case of equal first eigenvalue in the cells, $\mu_1 = \mu_2$. In the following we choose to normalize the first periodic eigenfunctions as follows:

$$\psi_1(0) = \psi_2(0) = 1. \tag{1.7}$$

We introduce a so-called discontinuity constant, α, defined by

$$\alpha = a_1(0)\frac{d\psi_1}{dy}(0) - a_2(0)\frac{d\psi_2}{dy}(0). \tag{1.8}$$

Note that $a_i \frac{d\psi_i}{dy}$ belongs to $H^1(\Omega_i)$ which is embedded in $C(\overline{\Omega}_i)$ (in 1-D) and therefore α is well defined as the trace of a continuous function at the origin. Three different situations are possible according to the sign of α.

If $\alpha = 0$, then the two periodic media are said to be well connected. In particular, the function equal to $a_i(d\psi_i)/(dx)$ in Ω_i is continuous through the interface (as well as ψ_i because of the normalization Condition (1.7)). Therefore, Theorem 1.1 extends easily to this case, and the discontinuity at the interface is not seen in the limit. Introducing a function $\psi(x/\epsilon) = \chi_1(x)\psi_1(x/\epsilon) + \chi_2(x)\psi_2(x/\epsilon)$, the eigenpairs $(\lambda_m^\epsilon, \phi_m^\epsilon)_{m\geq 1}$ satisfy

$$\lambda_m^\epsilon = \mu_1 + \epsilon^2 \nu_m + o(\epsilon^2) \text{ and } \phi_m^\epsilon(x) = u_m^\epsilon(x)\psi\left(\frac{x}{\epsilon}\right), \tag{1.9}$$

where u_m^ϵ converges weakly to u_m, and $(\lambda_m, u_m)_{m\geq 1}$ are the eigenpairs of the homogenized problem (see Theorem 3.1 and Figure 3.1)

$$\begin{cases} -\dfrac{d}{dx}\left(\left(\chi_1(x)\overline{D_1} + \chi_2(x)\overline{D_2}\right)\dfrac{du}{dx}\right) = \nu\left(\chi_1(x)\overline{\sigma_1} + \chi_2(x)\overline{\sigma_2}\right)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\alpha > 0$, the interface has a repelling effect, and each eigenfunction goes to 0 at the interface. The convergence result (1.9) still holds true, but the homogenized problem has an additional Dirichlet boundary condition at $x = 0$. More precisely, the limit homogenized problem is (see Theorem 3.1 and Figure 3.2):

$$\begin{cases} -\overline{D_1}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_1}u & \text{in } \Omega_1, \\ -\overline{D_2}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_2}u & \text{in } \Omega_2, \\ u = 0 & \text{on } \partial\Omega_1 \cup \partial\Omega_2. \end{cases}$$

If $\alpha < 0$, the situation is completely different since the first eigenfunction concentrates exponentially fast at the interface. In this latter case, there is no factorization principle as in Theorem 1.1, but rather a localization principle at the discontinuity (see Theorem 3.5 and Figure 3.3). The first eigenvalue λ_1^ϵ converges to a limit $0 < \lambda_1 < \mu_1 = \mu_2$, and $0 < \lambda_1^\epsilon - \lambda_1 < C\exp(-\tau/\epsilon)$, whereas the first normalized eigenvector satisfies

$$\left\| \frac{d}{dx}\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\frac{d}{dx}\left(\Psi\left(\frac{x}{\epsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\Psi\left(\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} \leq C\exp\left(-\frac{\tau}{\epsilon}\right).$$

The limit function $\Psi \in H^1(\mathbb{R})$ decreases exponentially away from the interface, since it is given by

$$\Psi(x) = \begin{cases} \psi_{1,\theta_1}(x) \text{ for } x < 0, \\ \psi_{2,\theta_2}(x) \text{ for } x > 0, \end{cases}$$

with $\lambda_1 = \mu_1(\theta_1) = \mu_2(\theta_2)$, and each of the eigenpairs $\left(\mu_i(\theta_i), \psi_{i,\theta_i}\right)$ being the first eigencouple of the following spectral cell problem:

$$\begin{cases} -\dfrac{d}{dx}\left(a_i(x)\dfrac{d\psi_{i,\theta_i}}{dx}\right) + \Sigma_i(x)\psi_{i,\theta_i} = \mu_i(\theta_i)\sigma_i(x)\psi_{i,\theta_i} \text{ in } [0,1], \\ x \to \psi_{i,\theta_i}(x)e^{-\theta_i x} \text{ 1-periodic.} \end{cases} \tag{1.10}$$

The required properties of the θ-parameterized family of spectral cell problems (1.10) are given in Section 2.

We now turn to the case $\mu_1 \neq \mu_2$, and with no loss of generality we assume $\mu_1 > \mu_2$. In this case too, the spectral cell problems (1.10) govern the limit behavior of (1.1). We introduce a positive parameter $\theta_0 > 0$, such that $\mu_1(\theta_0) = \mu_2$, and another discontinuity constant (see Lemma 3.9)

$$\alpha(\theta_0) = a_1(0)\frac{d\psi_{1,\theta_0}}{dy}(0) - a_2(0)\frac{d\psi_2}{dy}(0).$$

The sign of this new discontinuity constant determines the asymptotic behavior of (1.1).

If $\alpha(\theta_0) > 0$, the eigenfunctions ϕ_m^ϵ concentrate in the sub-domain Ω_2 where the first periodic eigenvalue is the smallest (see Theorem 3.8 in the simpler case when $\alpha \geq 0$, and Theorem 3.11 when $\alpha(\theta_0) > 0$). More precisely, the limit of ϕ_m^ϵ vanishes in the sub-domain Ω_1. Introducing the factorization $\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi_2(x/\epsilon)$ in Ω_2, the homogenized problem for the limit of u_m^ϵ is simply (see Figure 3.4)

$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_2}u & \text{in } \Omega_2, \\ u = 0 & \text{on } \partial\Omega_2. \end{cases}$$

The case $\alpha\,(\theta_0) = 0$ corresponds to the limit between localization at the interface and concentration in Ω_2. The limit of the eigenfunction ϕ_m^ϵ still vanishes in Ω_1, but in the homogenized problem the Dirichlet boundary condition at $x = 0$ is replaced by a Neumann boundary condition (see Theorem 3.12)

$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_2}u & \text{in } \Omega_2, \\ u(L) = 0 \text{ and } \dfrac{du}{dx}(0) = 0. & \end{cases}$$

Finally, when $\alpha\,(\theta_0) < 0$, a localization phenomenon appears, and the first eigenfunction concentrates exponentially fast at the interface. The result is then similar to the one obtained when $\mu_1 = \mu_2$ and $\alpha < 0$ (see Theorem 3.10).

Our main results are stated in Section 3, when μ_1 is equal to and not equal to μ_2. Before that, in Section 2, we give a few technical results on the spectral cell problems that are crucial not only for the proof, but also for the statement of our main results. Section 4 contains the proofs when the discontinuity constant is positive, $\alpha \geq 0$, while Section 5 focuses on the localization phenomena, namely $\alpha < 0$ or $\alpha(\theta_0) < 0$. Section 6 contains the proofs in the special situation when $\alpha < 0$ but no localization occurs ($\alpha(\theta_0) \geq 0$), as it can happen when μ_1 is not equal to μ_2. Section 7 contains the proof of a crucial technical result about the Hill equation in one dimension.

2. Cell problems

In order to state precisely our convergence results, the knowledge of the spectral cell problem (1.3) is not enough. As in [8], we need to introduce a parameterized family of spectral cell problems. They are reminiscent of the so-called Bloch wave decomposition (see e.g. [13], [24]), but they involve real exponentials instead of complex ones. All the results in this section are proved under the assumption that the periodic coefficients a_i, Σ_i, σ_i are positive, bounded, measurable functions, except Proposition 2.2 which asks for more smoothness or piecewise constant coefficients.

Lemma 2.1. *For each $\theta \in \mathbb{R}$ there exists a unique first eigencouple $(\psi_{i,\theta}, \mu_i(\theta))$, of the problem*

$$\begin{cases} -\dfrac{d}{dx}\left(a_i(x)\dfrac{d\psi_{i,\theta}}{dx}\right) + \Sigma_i(x)\psi_{i,\theta} = \mu_i(\theta)\sigma_i(x)\psi_{i,\theta} \text{ in } [0,1], \\ x \to \psi_{i,\theta}(x)e^{-\theta x} \quad \text{1-periodic and positive,} \end{cases} \tag{2.1}$$

which is normalized by

$$\psi_{i,\theta}(0) = 1. \tag{2.2}$$

The map $\theta \to \mu_i(\theta)$ is strictly concave with a maximum at $\theta = 0$, and satisfies the following inequalities:

$$c\theta^2 \leq \mu_i(0) - \mu_i(\theta) \leq C\theta^2,$$

where C and c are positive constants, independent of θ.

A further property of the first eigenfunction $\psi_{i,\theta}$, is given in the next proposition. Its proof is quite delicate and relies on purely 1-D arguments (we postpone it until Section 7). We give two different proofs: first, the case of C^2 coefficients, which allows us to perform a Liouville transformation and to use classical results on the 1-D Hill equation; second, the case of piecewise constant coefficients, which permits us to do explicit computations.

Proposition 2.2. *Assuming that the coefficients are C^2 or piecewise constant, for each $\theta \in \mathbb{R}$ the first eigenvector $\psi_{i,\theta}$ of Problem* (2.1) *with the normalization $\psi_{i,\theta}(0) = 1$ satisfies*

$$\lim_{\theta\to-\infty} \frac{d\psi_{i,\theta}}{dx}(0) = -\infty \quad \text{and} \quad \lim_{\theta\to+\infty} \frac{d\psi_{i,\theta}}{dx}(0) = +\infty.$$

Proof of Lemma 2.1. By introducing the change of variable

$$\phi_{i,\theta}(x) = \psi_{i,\theta}(x)e^{-\theta x},$$

Equation (2.1) is equivalent to

$$\begin{cases} -\dfrac{d}{dx}\left(a_i\dfrac{d\phi_{i,\theta}}{dx}\right) - \theta\left(\dfrac{d}{dx}\left(a_i\phi_{i,\theta}\right) + a_i\dfrac{d\phi_{i,\theta}}{dx}\right) \\ \qquad\qquad + \left(\Sigma_i - a_i\theta^2\right)\phi_{i,\theta} = \mu_i(\theta)\sigma_i\phi_{i,\theta} \text{ in } [0,1], \\ x \to \phi_{i,\theta}(x) \quad \text{1-periodic and positive,} \end{cases} \tag{2.3}$$

with the same normalization condition

$$\phi_{i,\theta}(0) = 1.$$

The existence of a unique first positive eigencouple for Problem (2.3) is known, see e.g. [15, Theorem 8.38], and we have $\phi_{i,\theta} \in H^1_\#([0,1]) \cap C^{0,s}([0,1])$, with $s > 0$. In particular, this implies that $C > \phi_{i,\theta}(x) > c > 0$ in $[0,1]$. It is proved in [8] that the function $\theta \to \mu_i(\theta)$ is smooth, strictly concave on all $\mathbb{R}$, and reaches its maximum at $\theta = 0$.

To obtain the growth condition on $\mu_i(\theta)$, we perform the following change of unknown:

$$u_\theta(x) = \frac{\psi_{i,\theta}(x)}{\psi_{i,0}(x)},$$

which is licit by virtue of Proposition 4.1. Then, u_θ is a solution of the following problem:

$$\begin{cases} -\dfrac{d}{dx}\left(b(x)\dfrac{du_\theta}{dx}\right) = \mu(\theta)s(x)u_\theta \quad \text{in } [0,1], \\ x \to u_\theta(x)e^{-\theta x} \quad \text{1-periodic}, \end{cases} \tag{2.4}$$

with $b(x) = a_i(x)\psi^2_{i,0}(x)$, $s(x) = \sigma_i(x)\psi^2_{i,0}(x)$, and $\mu(\theta) = \mu_i(\theta) - \mu_i(0)$. These coefficients are bounded, and we can therefore apply Lemma 2.3. □

Lemma 2.3. *Let b and s be measurable functions on* $[0,1]$*, bounded above and below by two positive constants $M > m > 0$. For each $\theta \in \mathbb{R}$ the first eigenvalue $\mu(\theta)$ of Problem* (2.4) *satisfies*

$$\frac{m}{M}\theta^2 \le -\mu(\theta) \le \frac{M}{m}\theta^2.$$

Proof. We already know that $\mu(\theta) < 0$ for all $\theta \neq 0$. We can assume that $\theta > 0$ since changing the sign of θ in (2.4) is equivalent to considering its adjoint equation which has the same first eigenvalue. Because we are working in one space dimension, (2.4) can be written as a system of ordinary differential equations. Namely, denoting by $'$ the x-derivation,

$$Y'(x) = A(x)Y(x) \text{ and } A = \begin{bmatrix} 0 & b^{-1} \\ -\mu(\theta)s & 0 \end{bmatrix} \text{ and } Y = \begin{pmatrix} Y_1 = u_\theta \\ Y_2 = bu'_\theta \end{pmatrix}. \tag{2.5}$$

By enforcing the normalization $u_\theta(0) = Y_1(0) = 1$, the Krein–Rutman Theorem implies that Y_1 is positive, and thus Y_2 is increasing. Since $Y_2(n) = e^{n\theta}Y_2(0)$, and $\theta > 0$, this implies that $Y_2(0) > 0$, and thus $Y_2(x) > 0$ for $x \ge 0$. This in turn gives, by the first equation, that Y_1 is increasing, thus $Y_1 \ge 1$ for $x \ge 0$. Because Y_1 and Y_2 are positive functions on $\mathbb{R}^+$, we can write

$$A^-Y \le Y' \le A^+Y \text{ with } A^+ = \begin{bmatrix} 0 & m^{-1} \\ -\mu(\theta)M & 0 \end{bmatrix}, \text{ and } A^- = \begin{bmatrix} 0 & M^{-1} \\ -\mu(\theta)m & 0 \end{bmatrix}.$$

Since the matrices A^+ and A^- have constant coefficients, it is straightforward to obtain the solutions of the initial value problems

$$Z' = -(A^-)^T Z,\ \ Z(0) = Z_0,\ \text{ and } X' = -(A^+)^T X,\ \ X(0) = X_0.$$

In particular, the choice $Z_0 = X_0 = \left(1, (-\mu(\theta)mM)^{-1/2}\right)$ leads to the positive solutions

$$Z(x) = Z_0 \exp\left(-x\sqrt{\frac{-\mu(\theta)m}{M}}\right) \text{ and } X(x) = X_0 \exp\left(-x\sqrt{\frac{-\mu(\theta)M}{m}}\right).$$

We can compute that $(Y \cdot Z)' = Y' \cdot Z + Y \cdot Z' = \left(Y' - A^- Y\right) \cdot Z \geq 0$ since Z is positive. Thus $Y \cdot Z \geq Y(0) \cdot Z(0)$ for all $x \geq 0$, and choosing $x = n \in \mathbb{N}$ leads to

$$Y(n) \cdot Z(n) = \exp\left(n\left(\theta - \sqrt{\frac{(-\mu(\theta))\, m}{M}}\right)\right) Y(0) \cdot Z(0) \geq Y(0) \cdot Z(0),$$

and therefore $\theta \geq \sqrt{\frac{(-\mu(\theta))m}{M}}$. Similarly, we have $(Y \cdot X)' = \left(Y' - A^+ Y\right) \cdot X \leq 0$ since X is positive, which in turn gives, for all n,

$$Y(n) \cdot X(n) = \exp\left(n\left(\theta - \sqrt{\frac{(-\mu(\theta))\, M}{m}}\right)\right) Y(0) \cdot X(0) \leq Y(0) \cdot X(0),$$

and therefore $\theta \leq \sqrt{\frac{(-\mu(\theta))M}{m}}$. □

Remark 2.4. Lemma 2.3 can be generalized to higher space dimensions by using the maximum principle. It is proved in [10] that, in general, $\theta \to \mu(\theta)$ is a strictly concave function, i.e., that on any bounded subset $K \subset \mathbb{R}^N$ (with N the space dimension), the Hessian matrix $H = \left(\frac{\partial^2 \mu}{\partial\theta_i \partial\theta_j}\right)_{1 \leq i,j \leq N}$ is negative definite and $Hx \cdot x \leq -C(K)x \cdot x$ with $C(K) > 0$. The function $\mu(\theta)$ achieves its maximum in 0 and $\lim_{|\theta| \to \infty} \mu(\theta) = -\infty$.

3. Main results

In the spirit of the method of proof of Theorem 1.1 (see [5]), we introduce in (1.1) the change of unknown

$$u^\epsilon(x) = \frac{\phi^\epsilon(x)}{\psi\left(x, \frac{x}{\epsilon}\right)},$$

with a function $\psi(x, y)$ defined by

$$\psi(x, y) = \chi_1(x)\psi_1(y) + \chi_2(x)\psi_2(y), \tag{3.1}$$

where (ψ_1, μ_1) and (ψ_2, μ_2) are the first eigencouples in each periodic cell of (1.3). By our normalization condition (1.7), the function $\psi(x, x/\epsilon)$ is continuous at the interface $x = 0$. On the contrary, the function $a(x, x/\epsilon)(d\psi(x, x/\epsilon))/(dx)$

is not necessarily continuous and its jump at the interface is measured by the discontinuity constant α introduced in (1.8).

The first result concerns the special case when the first cell eigenvalues of (1.3) are equal, $\mu_1 = \mu_2$, and the discontinuity constant is non-negative, $\alpha \geq 0$. Under these assumptions, we obtain a generalization of Theorem 1.1.

Theorem 3.1. *Let λ_m^ϵ and ϕ_m^ϵ be the m-th eigenvalue and normalized eigenvectors of* (1.1). *Assume that the discontinuity constant defined in* (1.8) *is non-negative $\alpha \geq 0$, and that $\mu_1 = \mu_2$. Then*

$$\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right) \quad \textit{and} \quad \lambda_m^\epsilon = \mu_1 + \epsilon^2 \nu_m + o(\epsilon^2),$$

up to a sub-sequence, u_m^ϵ converges weakly in $H_0^1(\Omega)$ towards u_m, and (ν_m, u_m) is the m-th eigencouple of the homogenized problem, which, if $\alpha = 0$, is

$$\begin{cases} -\dfrac{d}{dx}\left(\left(\chi_1(x)\overline{D_1} + \chi_2(x)\overline{D_2}\right)\dfrac{du}{dx}\right) = \nu\left(\chi_1(x)\overline{\sigma_1} + \chi_2(x)\overline{\sigma_2}\right)u & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.2}$$

and, if $\alpha > 0$, is

$$\begin{cases} -\overline{D_1}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_1}u & \textit{in } \Omega_1, \\ -\overline{D_2}\dfrac{d^2}{dx^2}u = \nu\overline{\sigma_2}u & \textit{in } \Omega_2, \\ u = 0 & \textit{on } \partial\Omega_1 \cup \partial\Omega_2. \end{cases} \tag{3.3}$$

In both cases, the homogenized coefficients are defined by Formula (1.5) *for each half domain.*

As an illustration of Theorem 3.1, we present some direct computations of the first eigenfunction ϕ_1^ϵ of Problem (1.1). The case $\mu_1 = \mu_2$ and $\alpha = 0$ is shown on Figure 3.1. (The domain is composed of a homogeneous medium on the left and an heterogeneous one on the right). The case $\mu_1 = \mu_2$ and $\alpha > 0$ is shown on Figure 3.2 (the domain is composed of two heterogeneous media with the same cell coefficients but with a constant phase shift between the right and the left). The data used for the computation is presented in Remark 3.7.

Remark 3.2. Of course, since the homogenized coefficients are constant in each sub-domain we can compute explicitly the eigenvalues of the homogenized problems in Theorem 3.1.

Remark 3.3. There is a simple sufficient condition for having well-connected media, i.e., $\alpha = 0$. If all coefficients satisfy a central symmetry condition, i.e., are symmetric with respect to the center of the unit cell [0, 1], then it is easy to check that ψ_i satisfies a Neumann boundary condition at $x = 0$ and $x = 1$, and therefore $\alpha = 0$. Actually, Theorem 3.1 was already proved by Malige [18] under this

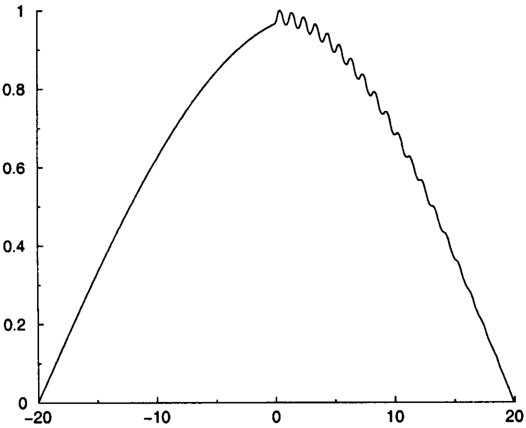

Fig. 3.1. First eigenfunction for Problem (1.1) in the case of two well-connected media, i.e., $\alpha = 0$

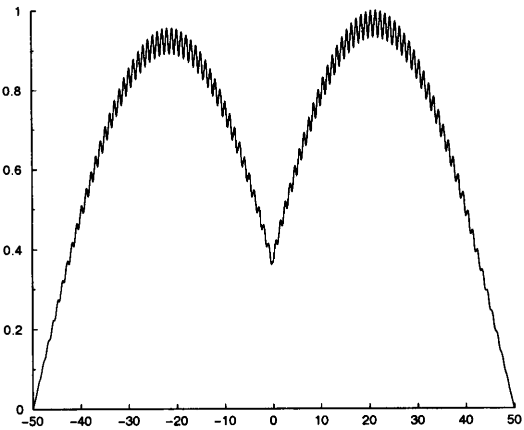

Fig. 3.2. First eigenfunction for Problem (1.1) in the case of non-well-connected media with a positive discontinuity constant $\alpha > 0$

assumption. The symmetry was used for the construction of the example shown in Figure 3.1: in Ω_2, the periodic coefficients are piecewise constant on $(0.3, 0.7)$ and $(0.1, 0.3) \cup (0.7, 1.0)$; in Ω_1, $a_1 \equiv 1$, $\sigma_1 \equiv 1$ and $\Sigma_1 = \mu_2$.

Remark 3.4. When $\alpha > 0$, the homogenized problem is posed on two disjoint sub-domains Ω_1 and Ω_2. In other words, there are two decoupled homogenized problems. Therefore, there always exist two non-negative eigenfunctions with disjoint supports, $u_1(x) = \sin(-\frac{\pi}{l}x)\chi_1(x)$ corresponding to the eigenvalue $\nu_1 = \pi^2 \frac{\overline{D_1}}{\overline{\sigma}_1 l^2}$ and $u_2(x) = \sin(\frac{\pi}{L}x)\chi_2(x)$ corresponding to the eigenvalue $\nu_2 = \pi^2 \frac{\overline{D_2}}{\overline{\sigma}_2 L^2}$. If the first eigenvalues in each sub-domain are distinct, e.g., $L^2\overline{\sigma_2}\overline{D_1} > l^2\overline{\sigma_1}\overline{D_2}$, the first factorized eigenfunction u_1^ϵ will tend to u_2, i.e., will concentrate in the sub-domain that has the smallest first eigenvalue and converge to zero in the other one. In the other case where the first eigenvalues in Ω_1 and Ω_2 are equal, the first eigen-subspace is of dimension 2, span by u_1 and u_2 and the uniqueness of the limit of ϕ_1^ϵ is lost (on Figure 3.2 the limit seems to be a linear combination of the first eigenfunctions on each sub-domain).

Our second result completes the case $\mu_1 = \mu_2$ when the discontinuity constant is negative, $\alpha < 0$. Under these assumptions, we obtain a localization phenomena.

Theorem 3.5. *Let* $(\lambda_1^\epsilon, \phi_1^\epsilon)$ *be the first normalized eigencouple of* (1.1). *Assume that* $\mu_1 = \mu_2$ *and* $\alpha < 0$. *Then, there exists a unique* $\lambda_1 > 0$ *and a unique positive* $\Psi(x) \in H^1(\mathbb{R})$ *such that*

$$0 \leq \lambda_1^\epsilon - \lambda_1 \leq C \exp\left(-\frac{\tau}{\epsilon}\right)$$

and

$$\left\| \frac{d}{dx}\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\frac{d}{dx}\left(\Psi\left(\frac{x}{\epsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\Psi\left(\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} \leq C \exp\left(-\frac{\tau}{\epsilon}\right),$$

where C *and* τ *are positive constants, independent of* ϵ. *The limit eigenvalue satisfies* $\lambda_1 < \mu_1 = \mu_2$, *and the limit eigenfunction is defined by*

$$\Psi(x) = \begin{cases} \psi_{1,\theta_1}(x) & \text{for } x < 0, \\ \psi_{2,\theta_2}(x) & \text{for } x > 0, \end{cases}$$

with $\theta_1 > 0$ *and* $\theta_2 < 0$, *and* $\left(\lambda_1, \psi_{i,\theta_i}\right)$ *is the first eigencouple of the cell problem* (2.1), *i.e.,*

$$\begin{cases} -\dfrac{d}{dx}\left(a_i(x)\dfrac{d\psi_{i,\theta_i}}{dx}\right) + \Sigma_i(x)\psi_{i,\theta_i} = \lambda_1\sigma_i(x)\psi_{i,\theta_i} & \text{in } [0,1], \\ x \to \psi_{i,\theta_i}(x)e^{-\theta_i x} \quad 1\text{-periodic}\ . \end{cases}$$

Remark 3.6. Theorem 3.5 is illustrated by Figure 3.3: the first eigenvector of system (1.1) converges exponentially fast towards a localized eigenfunction near the interface between the two domains. Furthermore, the corresponding eigenvalue is smaller than $\mu_1 = \mu_2$, which is the limit obtained in all the other cases. In contrast with Theorem 3.1, no factorization, or limit homogenized problem appears in the wording of Theorem 3.5. The limit eigenfunction Ψ contains both the periodical oscillations and the macroscopic trend.

Remark 3.7. The computations shown on Figure 3.2 and Figure 3.3 were performed with the same two media, but their positions are switched with respect to the interface when passing from one case to the other. We take $-l = L = 1$ with 100 periodicity cells, which yields $\epsilon = 0.02$. All the more the periodic cell coefficients for the two media are the same up to a phase shift in the unit cell. More precisely, in Figure 3.2 the coefficients are $a_1(y) = a(y)$, $a_2(y) = a(y+c)$, $\Sigma_1(y) = \Sigma(y)$, $\Sigma_2(y) = \Sigma(y+c)$, $\sigma_1(y) = \sigma(y)$, $\sigma_2(y) = \sigma(y+c)$, while in Figure 3.3 they are $a_1(y) = a(y+c)$, $a_2(y) = a(y)$, $\Sigma_1(y) = \Sigma(y+c)$, $\Sigma_2(y) = \Sigma(y)$, $\sigma_1(y) = \sigma(y+c)$, $\sigma_2(y) = \sigma(y)$, where $c = 0.6$ is a constant phase shift, and a, Σ and

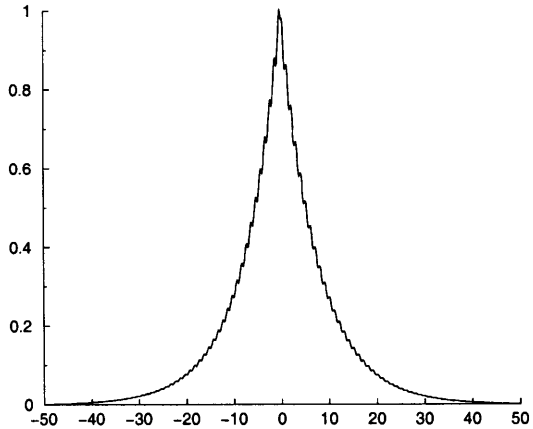

Fig. 3.3. First eigenfunction for Problem (1.1) in the case of non-well-connected media with a negative discontinuity constant $\alpha < 0$

σ are periodic functions. Each periodicity cell is made of three different media or constituents arranged in a specified order as follows:

$$(a, \Sigma, \sigma) = \begin{cases} (a_I, \Sigma_I, \sigma_I) & \text{if } 0 < y < 0.1 \\ (a_{II}, \Sigma_{II}, \sigma_{II}) & \text{if } 0.1 < y < 0.5 \\ (a_{III}, \Sigma_{III}, \sigma_{III}) & \text{if } 0.5 < y < 0.8 \\ (a_I, \Sigma_I, \sigma_I) & \text{if } 0.8 < y < 1, \end{cases}$$

with

Constituent *I*	$a_I = 0.9666$	$\Sigma_I = 2.1080$	$\sigma_I = 2.8283$
Constituent *II*	$a_{II} = 2.0086$	$\Sigma_{II} = 2.3878$	$\sigma_{II} = 2.9451$
Constituent *III*	$a_{III} = 2.0444$	$\Sigma_{III} = 2.9945$	$\sigma_{III} = 1.1493.$

Note that, by construction, $\mu_1 = \mu_2 \approx 1.3863$. The shape of the first eigenvector ϕ_1^ϵ on Figure 3.2 (with eigenvalue $\lambda_1^\epsilon \approx 1.3899$), corresponds to what is announced by Theorem 3.1: asymptotically, both media tend to separate when $\alpha > 0$. Therefore, by symmetry, Figure 3.3 corresponds to a situation where $\alpha < 0$: the first eigenvector concentrates exponentially at the interface between the two media. The numerical calculation confirms that the corresponding eigenvalue ($\lambda_1^\epsilon \approx 1.3720$) is below that of the periodicity cell. This phenomenon is explained by Lemma 5.4 which gives a necessary and sufficient condition for the existence of a localized eigensolution.

We now turn to the general case $\mu_1 \neq \mu_2$. In the following, we shall assume, without loss of generality, that

$$\mu_1 > \mu_2.$$

If the discontinuity constant is non-negative, i.e., $\alpha \geq 0$, the eigenfunctions concentrate asymptotically in the sub-domain Ω_2, where the first periodic eigenvalue is the smallest.

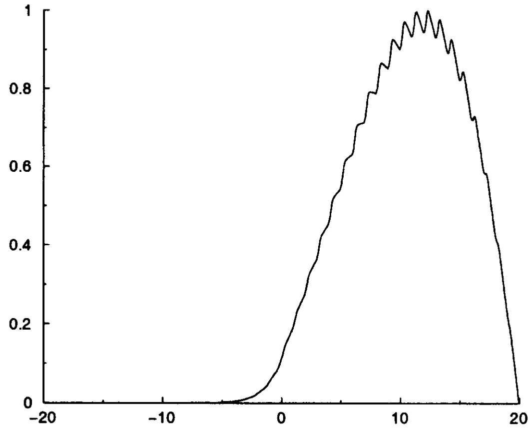

Fig. 3.4. First eigenfunction of (1.1) in the case of two media with $\mu_1 > \mu_2$ and $\alpha = 0$

Theorem 3.8. *Let λ_m^ϵ and ϕ_m^ϵ be the m-th eigenvalue and normalized eigenfunction of* (1.1). *Assume that $\alpha \geq 0$ and $\mu_1 > \mu_2$. Then,*

$$\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right) \qquad and \qquad \lambda_m^\epsilon = \mu_2 + \epsilon^2 \nu_m + o(\epsilon^2),$$

where, up to a sub-sequence, u_m^ϵ converges weakly in $H_0^1(\Omega)$ to u_m, with $u_m = 0$ in Ω_1 and (ν_m, u_m) is the m-th eigenpair of the following homogenized problem:

$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u_m = \nu_m \overline{\sigma_2} u_m & \text{in } \Omega_2, \\ u_m = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{3.4}$$

and the homogenized coefficients are still given by (1.5).

Figure 3.4 illustrates Theorem 3.8. It displays the first eigenfunction ϕ_1^ϵ in the case of two media with symmetric periodic structures (so that $\alpha = 0$), with $\mu_1 \simeq 1.58$ and $\mu_2 \simeq 0.43$, and 20 periodic cells on each side of the interface.

When $\mu_1 > \mu_2$, a localization phenomenon can also occur. Let us first remark that, as an obvious consequence of Lemma 2.1, we have the following result:

Lemma 3.9. *For all $\mu_1 > \mu_2$ there exists a unique $\theta_0 > 0$ such that $\mu_1(\theta_0) = \mu_2$.*

Indeed, Lemma 3.9 is obvious by remarking that $\mu_1(\theta)$, defined in Lemma 2.1, is a concave function with quadratic growth at infinity and reaching its maximum at $\theta = 0$, $\mu_1(0) = \mu_1 > \mu_2$. The first eigenvectors corresponding to μ_2 and $\mu_1(\theta_0)$ are denoted by ψ_2 and ψ_{1,θ_0}. They are continuous at the interface, i.e., $\psi_2(0) = \psi_{1,\theta_0}(0) = 1$, and we introduce a new discontinuity constant that will characterize the localization phenomenon

$$\alpha(\theta_0) = a_1(0)\frac{d\psi_{1,\theta_0}}{dy}(0) - a_2(0)\frac{d\psi_2}{dy}(0).$$

Theorem 3.10. *Let $(\lambda_1^\epsilon, \phi_1^\epsilon)$ be the first normalized eigencouple of* (1.1). *Assume that $\alpha(\theta_0) < 0$. Then, there exists a unique $\lambda_1 > 0$ and a unique positive $\Psi \in H^1(\mathbb{R})$ such that*

$$0 \le \lambda_1^\epsilon - \lambda_1 \le C \exp\left(-\frac{\tau}{\epsilon}\right)$$

and (3.5)

$$\left\| \frac{d}{dx}\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\frac{d}{dx}\left(\Psi\left(\frac{x}{\epsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\Psi\left(\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} \le C \exp\left(-\frac{\tau}{\epsilon}\right),$$

where C and τ are positive constants, independent of ϵ. The limit eigenvalue satisfies $\lambda_1 < \mu_2 < \mu_1$, and the limit eigenfunction is defined by

$$\Psi(x) = \begin{cases} \psi_{1,\theta_1}(x) & \text{for } x < 0, \\ \psi_{2,\theta_2}(x) & \text{for } x > 0, \end{cases}$$

with $\theta_1 > 0$ and $\theta_2 < 0$, and $(\lambda_1, \psi_{i,\theta_i})$ is the first eigencouple of the cell problem (2.1), *i.e.,*

$$\begin{cases} -\dfrac{d}{dx}\left(a_i(x)\dfrac{d\psi_{i,\theta_i}}{dx}\right) + \Sigma_i(x)\psi_{i,\theta_i} = \lambda_1\sigma_i(x)\psi_{i,\theta_i} & \text{in } [0,1], \\ x \to \psi_{i,\theta_i}(x)e^{-\theta_i x} & 1\text{-periodic.} \end{cases}$$

Finally, in the remaining case $\alpha < 0$ and $\alpha(\theta_0) \ge 0$, there is no localization, and the eigenfunctions still concentrate asymptotically in the sub-domain Ω_2 where the first periodic eigenvalue is the smallest. When $\alpha(\theta_0) > 0$, the limit problem has Dirichlet boundary conditions. When $\alpha(\theta_0) = 0$, the limit problem has a Neumann boundary condition at the interface.

Theorem 3.11. *Let λ_m^ϵ and ϕ_m^ϵ be the m-th eigenvalue and normalized eigenfunction of* (1.1). *Assume that $\mu_1 > \mu_2$, $\alpha < 0$ and $\alpha(\theta_0) > 0$. Then,*

$$\lambda_m^\epsilon = \mu_2 + \epsilon^2 \nu_m + o(\epsilon^2), \quad \phi_m^\epsilon(x) \to 0 \quad \text{in } L^2(\Omega_1)$$

and

$$\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi_2\left(\frac{x}{\epsilon}\right),$$

where, up to a sub-sequence, u_m^ϵ converges weakly in $H^1(\Omega_2)$ to u_m, and (ν_m, u_m) is the m-th eigenpair of the following homogenized problem:

$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u_m = \nu_m \overline{\sigma_2} u_m & \text{in } \Omega_2, \\ u_m = 0 & \text{on } \partial\Omega_2, \end{cases} \tag{3.6}$$

and the homogenized coefficients are still given by (1.5).

Theorem 3.12. *Let λ_m^ϵ and ϕ_m^ϵ be the m-th eigenvalue and normalized eigenfunction of* (1.1). *Assume that $\mu_1 > \mu_2$, $\alpha < 0$ and $\alpha(\theta_0) = 0$. Then,*

$$\lambda_m^\epsilon = \mu_2 + \epsilon^2 \nu_m + o(\epsilon^2), \quad \phi_m^\epsilon(x) \to 0 \quad \text{in } L^2(\Omega_1)$$

and

$$\phi_m^\epsilon(x) = u_m^\epsilon(x)\psi_2\left(\frac{x}{\epsilon}\right),$$

where, up to a sub-sequence, u_m^ϵ converges weakly in $H^1(\Omega_2)$ to u_m, and (ν_m, u_m) is the m-th eigenpair of the following homogenized problem:

$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u_m = \nu_m \overline{\sigma_2} u_m & \text{in } \Omega_2, \\ u_m(L) = 0, \ \text{and } \dfrac{du_m}{dx}(0) = 0, & \end{cases} \tag{3.7}$$

and the homogenized coefficients are still given by (1.5).

Remark 3.13. Note that the homogenized problems of Theorems 3.11 and 3.12 (corresponding to $\alpha(\theta_0) > 0$ and $\alpha(\theta_0) = 0$, respectively) are similar except the boundary condition at $x = 0$. Then a simple computation shows that the first homogenized eigenvalue ν_1 is four times smaller when $\alpha(\theta_0) = 0$ than when $\alpha(\theta_0) > 0$.

Remark 3.14. At the difference of Theorem 3.8, we do not prove in Theorems 3.11 and 3.12 that the factorized eigensolutions u_m^ϵ have bounded gradients in all of Ω, but simply within Ω_2. It is, therefore, difficult (at least for us) to study the possible occurrence of boundary layers in Ω_1. In the limit case of Theorem 3.12, because of the homogenized Neumann boundary condition at $x = 0$, we expect a non-trivial boundary layer in Ω_1.

Remark 3.15. The generalization of the results of this section to higher space dimensions is not obvious for at least two reasons. First, Theorems 3.5 and 3.10 rely on Proposition 2.2 which is proved only in one dimension (by using o.d.e. techniques). Second, even Theorems 3.1 and 3.8 (which do not depend on Proposition 2.2) are not straightforward in higher dimensions because we can not assume a perfect transmission condition (1.7) at the interface. Of course, if it happens by chance that, for a dimension $N > 1$, we have

$$\psi_{1,\theta_0}(0, y') = \psi_2(0, y') \tag{3.8}$$

for almost every $y' \in [0, 1]^{N-1}$, and

$$0 \le \alpha(y') = a_1(0, y')\frac{d\psi_{1,\theta_0}}{dy}(0, y') - a_2(0, y')\frac{d\psi_2}{dy}(0, y') \le M < +\infty, \tag{3.9}$$

then Theorems 3.1 and 3.8 extend easily since, at the interface, the problem is essentially one dimensional (see [9]). Of course, these conditions are very strict and almost never satisfied in practice. In general, we believe that boundary layers at the interface must be taken into account.

Remark 3.16. Throughout this paper we assume that, after rescaling by ϵ, the periodicity on both sides of the interface is exactly one. The fact that the period is the same in Ω_1 and Ω_2 is not important, and this is purely for convenience that we made this choice. All our results apply if the two periods are different, provided that the discontinuity constants α and $\alpha(\theta_0)$ are properly defined.

4. Proofs in the case $\alpha \geq 0$

In order to prove Theorems 3.1 and 3.8, we first need to justify the factorization $\phi^\epsilon(x) = u^\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)$. This is the goal of the next Proposition which is a generalization of a previous result of [5] (see also [3]).

Proposition 4.1. *Let* $\psi(x, y)$ *be the function defined by* (3.1). *Then, the linear operator* T *defined by*

$$\begin{aligned} T : H_0^1(\Omega) &\to \quad H_0^1(\Omega) \\ \phi(x) \quad &\to \frac{\phi(x)}{\psi\left(x, \frac{x}{\epsilon}\right)} \end{aligned}$$

is bounded, invertible and bicontinuous.

Proof. Thanks to the normalization condition (1.7) the function $\psi(x, x/\epsilon)$ is continuous on $\mathbb{R}$. By virtue of Lemma 2.1 we know that there exist two positive constants $C > c > 0$ such that $C \geq \psi_1(y), \psi_2(y) \geq c$ for all $y \in [0, 1]$, and these bounds also hold for ψ. Therefore, for all $\phi \in H_0^1(\Omega)$, if we define $u = T(\phi)$, we have

$$C^{-1} \|\phi\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} \leq c^{-1} \|\phi\|_{L^2(\Omega)}, \tag{4.1}$$

and T is an homeomorphism on $L^2(\Omega)$. On the other hand,

$$\begin{aligned} \int_\Omega a\frac{d\phi}{dx}\frac{d\phi}{dx} &= \int_{\Omega_1} a_1\psi_1^2\frac{du}{dx}\frac{du}{dx} + \int_{\Omega_2} a_2\psi_2^2\frac{du}{dx}\frac{du}{dx} \\ &\quad + \int_{\Omega_1} a_1\frac{d\psi_1}{dx}\frac{d(u^2\psi_1)}{dx} + \int_{\Omega_2} a_2\frac{d\psi_2}{dx}\frac{d(u^2\psi_2)}{dx}. \end{aligned} \tag{4.2}$$

Equation (1.3) defining $\psi_1(x/\epsilon)$ tested against $u^2(x)\psi_1(x/\epsilon)$ can be written

$$\int_{\Omega_1} a_1\frac{d\psi_1}{dx}\frac{d(u^2\psi_1)}{dx} - \frac{1}{\epsilon}a_1(0)\frac{d\psi_1}{dy}(0)u^2(0) = \frac{1}{\epsilon^2}\left(\mu_1\int_{\Omega_1}\sigma_1\psi_1^2u^2 - \int_{\Omega_1}\Sigma_1\psi_1^2u^2\right), \tag{4.3}$$

and similarly we have

$$\int_{\Omega_2} a_2\frac{d\psi_2}{dx}\frac{d(u^2\psi_2)}{dx} + \frac{1}{\epsilon}a_2(0)\frac{d\psi_2}{dy}(0)u^2(0) = \frac{1}{\epsilon^2}\left(\mu_2\int_{\Omega_2}\sigma_2\psi_2^2u^2 - \int_{\Omega_2}\Sigma_2\psi_2^2u^2\right). \tag{4.4}$$

When we replace (4.3) and (4.4) in (4.2) we obtain

$$\begin{aligned}\int_\Omega a\left(x,\frac{x}{\epsilon}\right)\frac{d\phi}{dx}\frac{d\phi}{dx}dx+\frac{1}{\epsilon^2}\int_\Omega \Sigma\left(x,\frac{x}{\epsilon}\right)\phi^2\,dx \\ =\sum_{i=1}^{2}\int_{\Omega_i} a_i\left(\frac{x}{\epsilon}\right)\psi_i^2\left(\frac{x}{\epsilon}\right)\frac{du}{dx}\frac{du}{dx}dx \\ +\frac{1}{\epsilon^2}\sum_{i=1}^{2}\int_{\Omega_i}\mu_i\sigma_i\left(\frac{x}{\epsilon}\right)\psi_i^2\left(\frac{x}{\epsilon}\right)u^2\,dx \\ +\frac{1}{\epsilon}\alpha\, u^2(0),\end{aligned} \tag{4.5}$$

where α is the discontinuity constant given by (1.8). If $\alpha \geq 0$, all the left-hand side terms are non-negative in (4.5). Since a_1, a_2, ψ_1 and ψ_2 are bounded below by positive constants, we can deduce that

$$\left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\leq C(\epsilon)\left(\left\|\frac{d\phi}{dx}\right\|^2_{L^2(\Omega)}+\|\phi\|^2_{L^2(\Omega)}\right). \tag{4.6}$$

Conversely, we have

$$0\leq \alpha\, u^2(0)\leq C\int_\Omega\left(\frac{du}{dx}\right)^2 dx, \tag{4.7}$$

therefore we also obtain from (4.5) that

$$\left\|\frac{d\phi}{dx}\right\|^2_{L^2(\Omega)}+\|\phi\|^2_{L^2(\Omega)}\leq C(\epsilon)\left(\left\|\frac{du}{dx}\right\|^2_{L^2(\Omega)}+\|u\|^2_{L^2(\Omega)}\right), \tag{4.8}$$

and this concludes the proof of the proposition for $\alpha \geq 0$. If $\alpha \leq 0$, note that thanks to the normalization condition (1.7), $u^2(0) = \phi^2(0)$. Consequently, identity (4.5) is also

$$\begin{aligned}\int_\Omega a\left(x,\frac{x}{\epsilon}\right)\left(\frac{d\phi}{dx}\right)^2 dx+\frac{1}{\epsilon^2}\int_\Omega \Sigma\left(x,\frac{x}{\epsilon}\right)\phi^2\,dx-\frac{1}{\epsilon}\alpha\phi^2(0) \\ =\sum_{i=1}^{2}\int_{\Omega_i} a_i\left(\frac{x}{\epsilon}\right)\psi_i^2\left(\frac{x}{\epsilon}\right)\left(\frac{du}{dx}\right)^2 dx+\frac{1}{\epsilon^2}\sum_{i=1}^{2}\int_{\Omega_i}\mu_i\sigma_i\left(\frac{x}{\epsilon}\right)\psi_i^2\left(\frac{x}{\epsilon}\right)u^2\,dx.\end{aligned}$$

We therefore obtain the same conclusion, reversing the positions of u and ϕ in (4.6–4.8). □

If we proceed to the change of unknown $u^\epsilon = T(\phi^\epsilon)$, Problem (1.1) is transformed into a new eigenvalue problem, where the singular perturbation in front of the divergence term has disappeared. Proposition 4.2 gives the form of this new problem after some simple algebra.

Proposition 4.2. *Introducing* $u^\epsilon(x) = \phi^\epsilon(x)/\psi\left(x, \frac{x}{\epsilon}\right)$, (1.1) *is equivalent to the following eigenvalue problem:*

$$\begin{cases} -\dfrac{d}{dx}\left(D\left(x, \dfrac{x}{\epsilon}\right)\dfrac{du^\epsilon}{dx}\right) + \dfrac{\mu_1 - \mu_2}{\epsilon^2}\chi_1(x)B\left(x, \dfrac{x}{\epsilon}\right)u^\epsilon & \\ \qquad + \epsilon^{-1}\alpha u^\epsilon(0)\delta(x) = \nu^\epsilon B\left(x, \dfrac{x}{\epsilon}\right)u^\epsilon & \text{in } \Omega \\ u^\epsilon = 0 \quad \text{on } \partial\Omega, & \end{cases} \tag{4.9}$$

where $\delta(x)$ *is the Dirac function, the (positive) diffusion coefficient is defined by*

$$D\left(x, \frac{x}{\epsilon}\right) = \chi_1(x)\psi_1^2\left(\frac{x}{\epsilon}\right)a_1\left(\frac{x}{\epsilon}\right) + \chi_2(x)\psi_2^2\left(\frac{x}{\epsilon}\right)a_2\left(\frac{x}{\epsilon}\right),$$

the (positive) coefficient B by

$$B\left(x, \frac{x}{\epsilon}\right) = \chi_1(x)\psi_1^2\left(\frac{x}{\epsilon}\right)\sigma_1\left(\frac{x}{\epsilon}\right) + \chi_2(x)\psi_2^2\left(\frac{x}{\epsilon}\right)\sigma_2\left(\frac{x}{\epsilon}\right),$$

and the new eigenvalue by

$$\nu^\epsilon = \frac{\lambda^\epsilon - \mu_2}{\epsilon^2}.$$

Remark 4.3. We proved Proposition 4.1 regardless of the sign of α, therefore Proposition 4.2 is also valid when $\alpha < 0$. We shall use this equivalent form of (1.1) in Section 6.

Following a strategy already used in [3], [4], the asymptotic study of the eigenvalue problem (4.9) relies on the detailed homogenization, as ϵ tends to zero, of the following problem:

$$\begin{cases} -\dfrac{d}{dx}\left(D\left(x, \dfrac{x}{\epsilon}\right)\dfrac{du^\epsilon}{dx}\right) + \dfrac{\mu_1 - \mu_2}{\epsilon^2}\chi_1(x)B\left(x, \dfrac{x}{\epsilon}\right)u^\epsilon & \\ \qquad + \epsilon^{-1}\alpha u^\epsilon(0)\delta(x) = f_\epsilon & \text{in } \Omega \\ u^\epsilon = 0 \text{ on } \partial\Omega, & \end{cases} \tag{4.10}$$

with a right-hand side f_ϵ, which is a bounded sequence of $L^2(\Omega)$, weakly converging to a limit $f \in L^2(\Omega)$. We first obtain a priori estimates:

Proposition 4.4. *If* $\alpha \geq 0$*, the solution* u^ϵ *of Equation* (4.10) *satisfies*

$$\|u^\epsilon\|_{H_0^1(\Omega)} + \frac{\mu_1 - \mu_2}{\epsilon}\|u^\epsilon\|_{L^2(\Omega_1)} + \sqrt{\frac{\alpha}{\epsilon}}\|u^\epsilon(0)\| \leq C\|f_\epsilon\|_{L^2(\Omega)}, \tag{4.11}$$

where C *is a constant independent of* ϵ*. Therefore, up to a sub-sequence,* u^ϵ *converges weakly to a limit* u *in* $H_0^1(\Omega)$*. Furthermore,*

- *if $\mu_1 > \mu_2$, the limit u vanishes in Ω_1 and thus belongs to $H_0^1(\Omega_2)$;*
- *if $\mu_1 = \mu_2$ and $\alpha > 0$, the limit satisfies $u(0) = 0$ and thus can be written $u = u_1 + u_2$ with $u_1 \in H_0^1(\Omega_1)$ and $u_2 \in H_0^1(\Omega_2)$.*

Proof. If we variationally test Equation (4.10) defining u^ϵ against u^ϵ, we obtain

$$\int_\Omega D\left(x, \frac{x}{\epsilon}\right)\left(\frac{du^\epsilon}{dx}\right)^2 dx + \frac{\mu_1 - \mu_2}{\epsilon^2}\int_{\Omega_1} B\left(x, \frac{x}{\epsilon}\right)(u^\epsilon)^2\, dx + \frac{1}{\epsilon}\alpha(u^\epsilon)^2(0) = \int_\Omega f_\epsilon u^\epsilon\, dx.$$

Since D and B are bounded below by a positive constant, and since we assume that $\mu_1 \geq \mu_2$ and $\alpha \geq 0$, we obtain

$$\left\|\frac{du^\epsilon}{dx}\right\|^2_{L^2(\Omega)} + \frac{\mu_1 - \mu_2}{\epsilon^2}\|u^\epsilon\|^2_{L^2(\Omega_1)} + \frac{\alpha}{\epsilon}\|u^\epsilon(0)\|^2 \leq C\,\|f_\epsilon\|_{L^2(\Omega)}\,\|u^\epsilon\|_{L^2(\Omega)},$$

which yields the desired result thanks to the Poincaré inequality.

Lemma 4.5. *Let S_ϵ be the operator defined by*

$$\begin{aligned} S_\epsilon : L^2(\Omega) &\to L^2(\Omega) \\ f &\to u^\epsilon \text{ unique solution in } H_0^1(\Omega) \\ &\quad \text{of Equation (4.10) with r.h.s. } f. \end{aligned} \tag{4.12}$$

For all fixed $\epsilon > 0$, S_ϵ is a linear compact operator in $L^2(\Omega)$.

This result is a consequence of the a priori estimate (4.11) and of the compact inclusion of $H_0^1(\Omega)$ in $L^2(\Omega)$. We shall show the following result:

Proposition 4.6. *Let f_ϵ be a weakly converging sequence to a limit f in $L^2(\Omega)$. The sequence $u^\epsilon = S_\epsilon(f_\epsilon)$ weakly converges in $H_0^1(\Omega)$ towards u^0 defined by $u^0 = S(f)$.*

1. *If $\alpha = 0$ and $\mu_1 = \mu_2$, then S is the following compact operator:*

$$\begin{aligned} S : L^2(\Omega) &\to L^2(\Omega) \\ f &\to u \text{ unique solution of} \end{aligned}$$
$$\begin{cases} -\dfrac{d}{dx}\left(\left(\chi_1(x)\overline{D_1} + \chi_2(x)\overline{D_2}\right)\dfrac{d}{dx}u(x)\right) = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\overline{D_1}$ and $\overline{D_2}$ are given by (1.5).

2. *If $\alpha \geq 0$ and $\mu_1 > \mu_2$, then S is the following compact operator:*

$$\begin{aligned} S : L^2(\Omega) &\to L^2(\Omega) \\ f &\to u \text{ unique solution of} \end{aligned}$$
$$\begin{cases} -\overline{D_2}\dfrac{d^2}{dx^2}u(x) = f & \text{in } \Omega_2, \\ u = 0 \quad \text{on } \overline{\Omega} \setminus \Omega_2. \end{cases}$$

3. *If $\alpha > 0$ and $\mu_1 = \mu_2$, then S is the following compact operator:*

$$\begin{aligned} S : L^2(\Omega) &\to L^2(\Omega) \\ f \quad &\to u \text{ unique solution of} \\ &\begin{cases} -\overline{D_1}\dfrac{d^2}{dx^2}u(x) = f & \text{in } \Omega_1, \\ -\overline{D_2}\dfrac{d^2}{dx^2}u(x) = f & \text{in } \Omega_2, \\ u = 0 \quad \text{on } \partial\Omega_2 \cup \partial\Omega_2. \end{cases} \end{aligned}$$

Proof. The proof is quite standard in homogenization theory. For example, using the notion of two-scale convergence (see [1], [22]) it is an easy exercise that we safely leave to the reader (the details can be found in [9] if necessary). Let us simply remark that, if $\mu_1 = \mu_2$ and $\alpha = 0$, then the homogenization of (4.10) is completely obvious. If $\mu_1 = \mu_2$ and $\alpha > 0$, then the a priori estimates of Proposition 4.4 show that $u^\epsilon(0)$ goes to zero, while, if $\mu_1 > \mu_2$ and $\alpha \geq 0$, they imply that u^ϵ goes to zero in Ω_1. □

We are now able to conclude the proofs of Theorems 3.1 and 3.8.

Proof of Theorems 3.1 and 3.8. Let us first remark that Proposition 4.6 implies that the sequence of operators S_ϵ, defined by (4.12), uniformly converges to the limit operator S. The asymptotic analysis of the eigenvalue problem (4.9) is truly given by that of T_ϵ, given by,

$$\begin{aligned} T_\epsilon : L^2(\Omega) &\to L^2(\Omega) \\ f \qquad &\to S_\epsilon\left(B\left(x, \frac{x}{\epsilon}\right) f\right). \end{aligned}$$

The eigenvalues of T_ϵ being the inverse of that of (4.9). Introducing $\overline{\sigma}(x) = \int_0^1 B(x, y)dy$, which is the weak limit of $B(x, \frac{x}{\epsilon})$, we define the limit operator T by

$$\begin{aligned} T : L^2(\Omega) &\to L^2(\Omega) \\ f \qquad &\to S\left(\overline{\sigma} f\right). \end{aligned}$$

The sequence T_ϵ does not uniformly converge to T, but the sequence T_ϵ is nevertheless sequentially compact, in the sense that

$$\begin{cases} \forall f \in L^2(\Omega) \quad \lim_{\epsilon\to 0} \|T_\epsilon(f) - T(f)\|_{L^2(\Omega)} = 0, \\ \text{the set } \{T_\epsilon(f), \ \|f\|_{L^2(\Omega)} \leq 1, \epsilon \geq 0\} \text{ is sequentially compact.} \end{cases}$$

Theorems 3.1 and 3.8 are then consequences of Theorem 4.7 (see also Chapter 11 in [16]). □

Theorem 4.7. (see e.g. [7], [12]) *Let T_n be a sequence of compact operators that converges to T. Assume that $(T_n)_{n\geq 1}$ is collectively compact and T is compact. Let $\mu \in \mathbb{C}$ be an eigenvalue of T, of multiplicity m. Let Γ be a smooth curve enclosing μ in the complex plane and leaving outside the rest of the spectrum of T. Then, for sufficiently large values of n, Γ also encloses exactly m eigenvalues of T_n and leaves outside the rest of the spectrum of T_n.*

5. Proofs in the case $\alpha(\theta_0) < 0$

The goal of this section is to prove Theorems 3.5 and 3.10. To understand the asymptotic behavior of problem (1.1) when the discontinuity constant $\alpha(\theta_0)$ is negative, we first rescale the equations by introducing the change of variables $y = \frac{x}{\epsilon}$. Then, problem (1.1) is equivalent to

$$\begin{cases} -\dfrac{d}{dy}\left(a(y)\dfrac{d\varphi^\epsilon}{dy}\right) + \Sigma(y)\varphi^\epsilon = \lambda^\epsilon\sigma(y)\varphi^\epsilon \text{ in } \Omega_\epsilon, \\ \varphi^\epsilon(-\epsilon^{-1}l) = \varphi^\epsilon(\epsilon^{-1}L) = 0, \end{cases} \tag{5.1}$$

with $\Omega_\epsilon =]-\epsilon^{-1}l, \epsilon^{-1}L[$, $\varphi^\epsilon(y) = \phi^\epsilon(\frac{x}{\epsilon})$, and

$$a(y)\ (\Sigma(y), \sigma(y),\ \text{respectively}) = \begin{cases} a_1(y)\ (\Sigma_1(y), \sigma_1(y),\ \text{respectively}) & \text{if } y < 0, \\ a_2(y)\ (\Sigma_2(y), \sigma_2(y),\ \text{respectively}) & \text{if } y > 0. \end{cases}$$

As ϵ goes to 0, the domain Ω_ϵ converges to $\mathbb{R}$, and formally the limit problem of (5.1) is

$$\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{d\Psi}{dx}\right) + \Sigma(x)\Psi = \lambda\sigma(x)\Psi \text{ in } \mathbb{R}, \\ \Psi \in H^1(\mathbb{R}). \end{cases} \tag{5.2}$$

We first recall some properties of the spectrum of (5.2). We introduce the Green operator S acting in $L^2(\mathbb{R})$ defined by

$$\begin{aligned} S : L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ f &\to u \text{ unique solution in } H^1(\mathbb{R}) \text{ of} \\ &-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right) + \Sigma(x)u = \sigma(x)f \text{ in } \mathbb{R}. \end{aligned} \tag{5.3}$$

The eigenvalues of S are precisely the inverse of those of (5.2). Nevertheless, to simplify the discussion we shall say that λ is an eigenvalue of S, or (5.2), if its inverse belongs to the spectrum of S.

Proposition 5.1. *The operator S is self-adjoint and non-compact. Its spectrum can be decomposed in its discrete and essential part, $\sigma(S) = \sigma_{disc}(S) \cup \sigma_{ess}(S)$. The lower bound of the essential spectrum is equal to the smallest cell first eigenvalue in* (1.3)*, namely*

$$\min \sigma_{ess}(S) = \min(\mu_1, \mu_2).$$

If (λ, Ψ) is an eigencouple in the discrete spectrum, then there exist $\theta_1 > 0$ and $\theta_2 < 0$ such that

$$\Psi(x) = \begin{cases} \psi_1(x) & \text{if } x < 0, \\ \psi_2(x) & \text{if } x > 0, \end{cases}$$

and (λ, ψ_i) *is an eigencouple of*

$$\begin{cases} -\dfrac{d}{dx}\left(a_i(x)\dfrac{d\psi_i}{dx}\right) + \Sigma_i(x)\psi_i = \lambda\sigma_i(x)\psi_i \textit{ in } [0, 1], \\ x \to \psi_i(x)e^{-\theta_i x} \text{ 1-}\textit{periodic}. \end{cases} \tag{5.4}$$

Remark 5.2. By definition, the discrete spectrum of S is composed of isolated eigenvalues of finite multiplicity, while its essential spectrum is characterized by the Weyl criterion, i.e., $\forall\lambda \in \sigma_{ess}(S)$ there exists a sequence $\{u_n\} \in L^2(\mathbb{R})$ such that

$$\begin{cases} \|u_n\|_{L^2(\mathbb{R})} = 1, \ u_n \to 0 \text{ in } L^2(\mathbb{R}) \text{ weakly}, \\ (S - \lambda Id)u_n \to 0 \text{ in } L^2(\mathbb{R}) \text{ strongly}. \end{cases}$$

Proposition 5.1 tells us in particular that $\sigma_{ess}(S)$ is not empty and that any discrete eigenvector decays exponentially at infinity. Remark that Equation (5.4) is similar to (2.1).

Proof. The study of the spectrum of S is classical. The exponential decay of the discrete eigenfunctions is obtained through Floquet Theory (see, e.g., [20], [24]). The same tool yields the lower bound of the essential spectrum (see [4], [13]). Note that these results are obtained under the mere assumption that the coefficients of Equation (5.3) are positive measurable functions (no smoothness is required). □

In order to pass to the limit $\epsilon \to 0$ in (5.1), we also introduce an operator S_ϵ acting in $L^2(\mathbb{R})$ defined by

$$\begin{aligned} S_\epsilon : L^2(\mathbb{R}) &\to L^2(\mathbb{R}) \\ f &\to u^\epsilon \text{ unique solution in } H_0^1(\Omega_\epsilon) \text{ of} \\ &\begin{cases} -\dfrac{d}{dx}\left(a(x)\dfrac{du^\epsilon}{dx}\right) + \Sigma(x)u^\epsilon = \sigma(x)f, \text{ in } \Omega_\epsilon \\ u^\epsilon(x) = 0 \text{ on } \partial\Omega_\epsilon. \end{cases} \end{aligned} \tag{5.5}$$

The operator S_ϵ is compact and its eigenvalues are the inverses of that of (5.1). Unfortunately, the convergence of the sequence S_ϵ to S is not uniform, so that the limit of the spectrum of S_ϵ is not the spectrum of S. Nevertheless, this limit can be characterized explicitly and we recall the following result that may be found in [4]:

Proposition 5.3. *For all* $f \in L^2(\mathbb{R})$*,* $S_\epsilon(f)$ *converges strongly to* $S(f)$ *in* $L^2(\mathbb{R})$*, and we have*

$$\lim_{\epsilon\to 0} \sigma\left(S_\epsilon\right) = \sigma(S) \cup \sigma_{BL}.$$

Furthermore, the first eigenvalue λ_1^ϵ *converges to a limit* λ_1 *which does belong to the spectrum of* S *and is thus the smallest element of* $\sigma(S)$*. We also have*

$$\min\sigma_{BL} = \min\sigma_{ess}(S) = \min\left(\mu_1, \mu_2\right). \tag{5.6}$$

That part of the limit spectrum, denoted by σ_{BL}, is called the boundary layer spectrum. It can be characterized completely in terms of an equation similar to (5.2) but in the half-line (for details, see [4]). We do not dwell on this boundary layer spectrum since we only need to know (5.6) in the following:

Lemma 5.4. *Let θ_0 be defined as in Lemma* 3.9*, i.e., $\mu_1(\theta_0) = \mu_2$, and ψ_{1,θ_0} the corresponding eigenvector defined by* (2.1)*. Let $\alpha(\theta_0)$ be defined by*

$$\alpha(\theta_0) = a_1(0)\frac{d\psi_{1,\theta_0}}{dy}(0) - a_2(0)\frac{d\psi_2}{dy}(0).$$

If and only if

$$\alpha(\theta_0) < 0, \tag{5.7}$$

the limit λ_1 of the first eigenvalue λ_1^ϵ of Problem (1.1) *satisfies*

$$\lambda_1 < min(\mu_1, \mu_2).$$

Remark 5.5. In particular, this lemma applies when $\mu_1 = \mu_2$, and $\alpha \equiv \alpha(\theta_0) < 0$. It implies that, when the discontinuity constant is negative, the limit first eigenvalue cannot be predicted by the homogenized models obtained under a strict periodicity assumption on each side of the interface. The proof of Lemma 5.4 relies on Proposition 2.2, that we have not been able to prove in the general case, but under the additional assumption that the coefficients are C^2, or piecewise constant.

Proof of Lemma 5.4. For all $\theta \in [\theta_0, +\infty[$, where θ_0 is defined in Lemma 3.9, because of the concavity of $\mu_1(\theta)$ and $\mu_2(\theta)$, we can associate to each θ a unique $\theta' \leq 0$ such that $\mu_1(\theta) = \mu_2(\theta')$ and $\psi_{2,\theta'}$ is the first eigenvector defined by

$$\begin{cases} -\dfrac{d}{dx}\left(a_2(x)\dfrac{d\psi_{2,\theta'}}{dx}\right) + \Sigma_2(x)\psi_{2,\theta'} = \mu_2(\theta')\sigma_2(x)\psi_{2,\theta'} \text{ in } [0,1] \\ x \to \psi_{2,\theta}(x)e^{-\theta' x} \text{ 1-periodic }, \\ \psi_{2,\theta'}(0) = 1. \end{cases} \tag{5.8}$$

Note that for $\theta = \theta_0$, we have $\theta' = 0$. The pair (λ, Ψ) defined by

$$\begin{aligned} \lambda &= \mu_1(\theta) = \mu_2(\theta') \\ \Psi &= \psi_{1,\theta} \text{ for } x > 0, \\ \Psi &= \psi_{2,\theta'} \text{ for } x < 0, \end{aligned}$$

is an eigencouple for Problem (5.2) if and only if

$$\alpha(\theta) = a_1(0)\frac{d\psi_{1,\theta}}{dx}(0) - a_2(0)\frac{d\psi_{2,\theta'}}{dx}(0) = 0. \tag{5.9}$$

Thanks to Proposition 2.2, we have

$$\lim_{\theta\to+\infty} \alpha(\theta) = \lim_{\theta\to+\infty} a_1(0)\frac{d\psi_{1,\theta}}{dx}(0) - \lim_{\theta'\to-\infty} a_2(0)\frac{d\psi_{2,\theta'}}{dx}(0) = +\infty.$$

Therefore, if we assume $\alpha(\theta_0) < 0$ then Equation (5.9) admits a solution, for some $\theta < \theta_0$, and $\theta' > 0$. We have thus obtained a value of θ such that $\lambda < \min(\mu_1, \mu_2)$. Finally, since $\lambda_1 \leq \lambda$, by virtue of Proposition 5.1, we have $\lambda_1 \in \sigma_{disc}(S)$.

Conversely, if $\lambda_1 < \min(\mu_1, \mu_2)$ we know from Proposition 5.6 that on both sides of the origin the corresponding eigenfunction Ψ has an exponential decay. Then Proposition 5.1 shows that it must of the form $\Psi = c\psi_{1,\theta}$ and $\Psi = c\psi_{2,\theta'}$ for some θ and θ' on each half line. Since Identity (5.9) is a necessary and sufficient condition for the existence of such a Ψ, and the proof is complete. □

Proof of Theorems 3.5 and 3.10. Thanks to Lemma 5.4 and Proposition 5.1, if Condition (5.7) is satisfied then λ_1 is in the discrete spectrum of Problem (5.2). Theorems 3.5 and 3.10 are then a consequence of Proposition 5.6. Indeed the eigenfunction $\phi_1^\epsilon(x)$ in Theorems 3.5 and 3.10 is equal to $\frac{1}{\sqrt{\epsilon}}\varphi_1^\epsilon\left(\frac{x}{\epsilon}\right)$, where φ_1^ϵ is the first eigenfunction in Proposition 5.6. Inequality (5.10) then becomes

$$\epsilon^2 \left\| \frac{d}{dx}\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\frac{d}{dx}\left(\Psi\left(\frac{x}{\epsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\Psi\left(\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} \leq C\exp\left(-\frac{\tau}{\epsilon}\right),$$

which in turn implies

$$\left\| \frac{d}{dx}\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\frac{d}{dx}\left(\Psi\left(\frac{x}{\epsilon}\right)\right)\right\|_{L^2(\Omega)} + \left\|\phi_1^\epsilon(x) - \frac{1}{\sqrt{\epsilon}}\Psi\left(\frac{x}{\epsilon}\right)\right\|_{L^2(\Omega)} \leq C'\exp\left(-\frac{\tau'}{\epsilon}\right),$$

for any $\tau' < \tau$. □

Proposition 5.6. *Assume that Problem* (5.2)*, or equivalently operator S, admits a first positive normalized eigencouple* (λ_1, Ψ) *such that* $\lambda_1 < \min(\mu_1, \mu_2)$*. Then the first positive normalized eigencouple* $\left(\lambda_1^\epsilon, \varphi_1^\epsilon\right)$ *of* (5.1)*, or of* S_ϵ*, satisfies*

$$0 \leq \lambda_1^\epsilon - \lambda_1 \leq C\exp\left(-\frac{\tau}{\epsilon}\right) \quad \text{and} \quad \left\|\frac{d}{dx}\varphi_1^\epsilon - \frac{d}{dx}\Psi\right\|_{L^2(\Omega_\epsilon)} + \left\|\varphi_1^\epsilon - \Psi\right\|_{L^2(\Omega_\epsilon)} \leq C\exp\left(-\frac{\tau}{\epsilon}\right), \qquad (5.10)$$

where C and τ *are strictly positive constant independent of* ϵ.

Proof. Since we assumed $\lambda_1 < \min\sigma_{ess}(S)$, we have

$$\lambda_1 = \min_{\substack{\varphi\in H^1(\mathbb{R})\\ \phi\neq 0}} \frac{\displaystyle\int_{\mathbb{R}} a(x)\left|\frac{d}{dx}\varphi\right|^2 dx + \int_{\mathbb{R}} \Sigma(x)\,\phi^2 dx}{\displaystyle\int_{\mathbb{R}} \sigma(x)\,\varphi^2 dx},$$

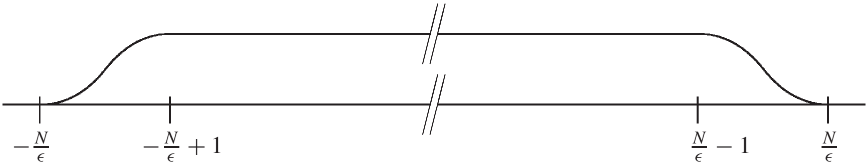

Fig. 5.1. Cut-off function χ

and this minimum is attained for $\varphi = \Psi$, which belongs to the discrete spectrum of S. We also have

$$\lambda_1^\epsilon = \min_{\substack{\varphi \in H_0^1(\Omega_\epsilon) \\ \phi \neq 0}} \frac{\displaystyle\int_{\Omega_\epsilon} a(x) \left| \frac{d}{dx}\varphi \right|^2 dx + \int_{\Omega_\epsilon} \Sigma(x)\, \varphi^2 dx}{\displaystyle\int_{\Omega_\epsilon} \sigma(x)\, \varphi^2 dx},$$

and this implies, by the inclusion of spaces that $\lambda_1 \leq \lambda_1^\epsilon$. Let χ be a smooth cut-off function, vanishing outside $\Omega_\epsilon = \left]-\frac{l}{\epsilon}, \frac{L}{\epsilon}\right[$, equal to 1 on $\left]-\frac{l}{\epsilon}+1, \frac{L}{\epsilon}-1\right[$, such that $0 \leq \chi \leq 1$, and $\frac{d\chi}{dx}$ does not depend on ϵ (see Figure 5.1). We then have $\chi\Psi \in H_0^1(\Omega_\epsilon)$, and

$$\lambda_1^\epsilon \leq \frac{\displaystyle\int_{\Omega_\epsilon} a(x) \left| \frac{d}{dx}(\chi\Psi) \right|^2 dx + \int_{\Omega_\epsilon} \Sigma(x)\, (\chi\Psi)^2 dx}{\displaystyle\int_{\Omega_\epsilon} \sigma(x)\, (\chi\Psi)^2 dx}. \tag{5.11}$$

By construction, $\frac{d\chi}{dx}$ has its support in $[-\frac{l}{\epsilon}, -\frac{l}{\epsilon}+1] \cup [\frac{L}{\epsilon}-1, \frac{L}{\epsilon}]$ and Inequality (5.11) becomes

$$\lambda_1^\epsilon \leq \frac{\displaystyle\int_{\mathbb{R}} a(x) \left| \frac{d}{dx}\Psi \right|^2 dx + \int_{\mathbb{R}} \Sigma(x)\, \Psi^2 dx + R_1^\epsilon}{\displaystyle\left(1 - R_2^\epsilon\right) \int_{\mathbb{R}} \sigma(x)\, \Psi^2 dx}, \tag{5.12}$$

with

$$R_1^\epsilon = 2\int_{\left[-\frac{l}{\epsilon}, -\frac{l}{\epsilon}+1\right] \cup \left[\frac{L}{\epsilon}-1, \frac{L}{\epsilon}\right]} a(x)\, |\Psi(x)| \left| \frac{d}{dx}\chi \right| \left(\left| \chi \frac{d}{dx}\Psi \right| + \left| \Psi \frac{d}{dx}\chi \right| \right) dx$$

and

$$R_2^\epsilon = \frac{\displaystyle\int_{-\infty}^{-\frac{l}{\epsilon}+1} \sigma(x)\Psi(x)^2 + \int_{\frac{L}{\epsilon}-1}^{+\infty} \sigma(x)\Psi(x)^2}{\displaystyle\int_{\mathbb{R}} \sigma(x)\Psi(x)^2}.$$

Thanks to Proposition 5.1, we know that

$$\sup_{x\in(-\infty,-\frac{l}{\epsilon})} |\Psi(x)| \le C\exp\left(-\theta_1\frac{l}{\epsilon}\right), \quad \text{and} \sup_{x\in(\frac{L}{\epsilon},+\infty)} |\Psi(x)| \le C\exp\left(\theta_2\frac{L}{\epsilon}\right)$$

with $\theta_1 > 0$ and $\theta_2 < 0$. We can deduce that $R_1^\epsilon \le C\exp\left(-\frac{\tau}{\epsilon}\right)$ and $R_2^\epsilon \le C\exp\left(-\frac{\tau}{\epsilon}\right)$ with $\tau = \min(l|\theta_1|, L|\theta_2|)$, and inserting these inequalities in (5.12) we obtain

$$\lambda_1^\epsilon \le \lambda_1\left(1 + C\exp\left(-\frac{\tau}{\epsilon}\right)\right). \tag{5.13}$$

Let us now show that φ_1^ϵ converges to Ψ. In order to obtain an approximation of Ψ that vanishes on the boundaries of the domain Ω_ϵ, we add to Ψ an affine function which compensates its values at both ends of the domain. We define $\Psi^\epsilon(x) = \Psi(x) + \ell^\epsilon(x)$, where ℓ^ϵ is the affine function such that

$$\Psi\left(-\frac{l}{\epsilon}\right) + \ell^\epsilon\left(-\frac{l}{\epsilon}\right) = 0 \quad \text{and} \quad \Psi\left(\frac{L}{\epsilon}\right) + \ell^\epsilon\left(\frac{L}{\epsilon}\right) = 0.$$

By construction, $\Psi^\epsilon \in H_0^1(\Omega_\epsilon)$, and Ψ^ϵ is a solution of the same problem than φ_1^ϵ up to a perturbation r^ϵ:

$$\begin{cases} -\dfrac{d}{dx}\left(a\,(x)\,\dfrac{d\Psi^\epsilon}{dx}\right) + \Sigma\,(x)\,\Psi^\epsilon = \lambda_1^\epsilon\sigma\,(x)\,\Psi^\epsilon + r^\epsilon \text{ in } \left]-\dfrac{l}{\epsilon}, \dfrac{L}{\epsilon}\right[\\ \Psi^\epsilon\left(-\dfrac{l}{\epsilon}\right) = \Psi^\epsilon\left(\dfrac{L}{\epsilon}\right) = 0. \end{cases} \tag{5.14}$$

The perturbation is $r^\epsilon = (\lambda_1 - \lambda_1^\epsilon)\sigma\Psi^\epsilon + \Sigma\ell^\epsilon - \lambda_1\sigma\ell^\epsilon - \frac{d}{dx}\left(a\frac{d\ell^\epsilon}{dx}\right) \in H^{-1}(\Omega_\epsilon)$. The coefficients being bounded, we obtain that for all $\phi \in H_0^1(\Omega_\epsilon)$,

$$\left|\int_{\Omega_\epsilon} r^\epsilon\phi\right| \le C\left(|\lambda - \lambda_1^\epsilon| + \sup_{\Omega_\epsilon}|\ell^\epsilon|\right)\|\phi\|_{L^2(\Omega_\epsilon)} + C\left\|\frac{d\ell^\epsilon}{dx}\right\|_{L^2(\Omega_\epsilon)}\left\|\frac{d\phi}{dx}\right\|_{L^2(\Omega_\epsilon)},$$

where C is a constant which does not depend on ϵ. From the exponential decay of Ψ we deduce that

$$\sup_{\Omega_\epsilon}|\ell^\epsilon| \le C\exp\left(-\frac{\tau}{\epsilon}\right) \quad \text{and} \quad \left\|\frac{d\ell^\epsilon}{dx}\right\|_{L^2(\Omega_\epsilon)} \le C\sqrt{\epsilon}\exp\left(-\frac{\tau}{\epsilon}\right), \tag{5.15}$$

and with the help of Estimate (5.13) we obtain

$$\left|\int_{\Omega_\epsilon} r^\epsilon\phi\right| \le C\exp\left(-\frac{\tau}{\epsilon}\right)\left(\|\phi\|_{L^2(\Omega_\epsilon)} + \left\|\frac{d\phi}{dx}\right\|_{L^2(\Omega_\epsilon)}\right). \tag{5.16}$$

The first eigenvalue λ_1^ϵ being simple, by a Fredholm alternative we can decompose Ψ^ϵ into a component proportional to φ_1^ϵ and a component orthogonal to φ_1^ϵ. We write $\Psi^\epsilon = \beta_\epsilon\phi_1^\epsilon + g^\epsilon$, where β_ϵ is a constant, and

$$\|g^\epsilon\|_{L^2(\Omega_\epsilon)} + \left\|\frac{dg^\epsilon}{dx}\right\|_{L^2(\Omega_\epsilon)} \le C_\epsilon\,\|r^\epsilon\|_{H^{-1}(\Omega_\epsilon)},$$

where C_ϵ is the norm of $\left(S_\epsilon^{-1} - \lambda_1^\epsilon Id\right)^{-1}$, a bounded operator defined on the orthogonal of the line generated by φ_1^ϵ. We have $C_\epsilon \le \frac{C}{|\lambda_1^\epsilon - \lambda_2^\epsilon|}$, where C is a constant independent of ϵ, and λ_2^ϵ is the next eigenvalue of S_ϵ. If we obtain that $\left|\lambda_1^\epsilon - \lambda_2^\epsilon\right| > c > 0$, with c independent of ϵ, we then deduce, with the help of Inequality (5.16),

$$\|g^\epsilon\|_{L^2(\Omega_\epsilon)} + \left\|\frac{dg^\epsilon}{dx}\right\|_{L^2(\Omega_\epsilon)} \le \frac{C}{c} \exp\left(-\frac{\tau}{\epsilon}\right). \tag{5.17}$$

From the decomposition $\Psi^\epsilon = \beta_\epsilon \varphi_1^\epsilon + g^\epsilon$, we get

$$|\beta_\epsilon| \left\|\varphi_1^\epsilon\right\|_{L^2(\Omega_\epsilon)} - \|g^\epsilon\|_{L^2(\Omega_\epsilon)} \le \|\Psi^\epsilon\|_{L^2(\Omega_\epsilon)} \le |\beta_\epsilon| \left\|\varphi_1^\epsilon\right\|_{L^2(\Omega_\epsilon)} + \|g^\epsilon\|_{L^2(\Omega_\epsilon)}.$$

We have $\left\|\varphi_1^\epsilon\right\|_{L^2(\Omega_\epsilon)} = 1$ and $\|\Psi\|_{L^2(\mathbb{R})} = 1$, thus

$$\begin{aligned}\left|\|\Psi^\epsilon\|_{L^2(\Omega_\epsilon)} - 1\right| = \left|\|\ell^\epsilon\|_{L^2(\Omega_\epsilon)} - \|\Psi\|_{L^2(\mathbb{R}\setminus\Omega_\epsilon)}\right| &\le \|\ell^\epsilon\|_{L^2(\Omega_\epsilon)} + \|\Psi\|_{L^2(\mathbb{R}\setminus\Omega_\epsilon)} \\ &\le C\frac{1}{\epsilon}\exp\left(-\frac{\tau}{\epsilon}\right)\end{aligned}$$

thanks to Estimate (5.15) and the exponential decay of Ψ. As a consequence, $||\beta_\epsilon| - 1| \le C\frac{1}{\epsilon}\exp\left(-\frac{\tau}{\epsilon}\right)$ and Ψ^ϵ and φ_1^ϵ being positives, we also have

$$|\beta_\epsilon - 1| \le C\frac{1}{\epsilon}\exp\left(-\frac{\tau}{\epsilon}\right). \tag{5.18}$$

Finally, if we write $\varphi_1^\epsilon(x) - \Psi(x) = (1-\beta_\epsilon)\varphi_1^\epsilon(x) - g^\epsilon(x) + \ell^\epsilon(x)$ on Ω_ϵ and using estimates (5.15), (5.17) and (5.18) we obtain

$$\left\|\frac{d\varphi_1^\epsilon}{dx} - \frac{d\Psi}{dx}\right\|_{L^2(\Omega_\epsilon)} + \left\|\varphi_1^\epsilon - \Psi\right\|_{L^2(\Omega_\epsilon)} \le C\frac{1}{\epsilon}\exp\left(-\frac{\tau}{\epsilon}\right) \le C\exp\left(-\frac{\tau'}{\epsilon}\right),$$

and this concludes the proof.

Let us now show that the spectral gap is uniformly bounded, i.e., $0 < c < \lambda_2^\epsilon - \lambda_1^\epsilon < C$. We know that λ_2^ϵ converges to a limit λ_2 which either belongs to $\sigma_{BL} \cup \sigma_{ess}(S)$ or to $\sigma_{disc}(S)$. In the latter case, the eigenvalues of the discrete spectrum are isolated so that $0 < c < \lambda_2 - \lambda_1 < C$. In the former case, we know from (5.6) that $\lambda_2 \ge \min(\mu_1, \mu_2)$, which is strictly larger than λ_1 by assumption, so that again $0 < c < \lambda_2 - \lambda_1 < C$. This yields the desired result for sufficiently small ϵ. □

6. Proofs in the case $\alpha < 0$ and $\alpha(\theta_0) \ge 0$

In this section we prove Theorems 3.11 and 3.12, following the strategy used in Section 4 for the case $\alpha \ge 0$. According to Proposition 4.2 and Remark 4.3, the original Problem (1.1) is equivalent to the factorized Problem (4.9) for any value of the discontinuity constant α. Introducing, as in Lemma 4.5, an operator S_ϵ, the convergence of (4.9) is governed by the homogenization of Problem (4.10) with a given right-hand side. The key element for the proof of Proposition 4.6, and in turn Theorem 3.8, is the a priori estimate given by Proposition 4.4. It does not hold for $\alpha < 0$. Nevertheless, the arguments of the proof of Proposition 4.4 yields a similar result that we state in Proposition 6.1 below:

Proposition 6.1. *The solution u^ϵ of Equation* (4.10) *satisfies*

$$\|u^\epsilon\|^2_{H^1_0(\Omega)} + \frac{\mu_1 - \mu_2}{\epsilon^2} \|u^\epsilon\|^2_{L^2(\Omega_1)} + \frac{\alpha}{\epsilon} |u^\epsilon(0)|^2 \le C \|f_\epsilon\|_{L^2(\Omega)} \|u^\epsilon\|_{L^2(\Omega)}, \quad (6.1)$$

where C is a constant independent of ϵ.

Since we assumed $\alpha < 0$, (6.1) alone does not furnish sufficient a priori estimates for concluding. Thus, for the proof of Theorems 3.11 and 3.12 we need an additional lemma.

Lemma 6.2. *Assume that $\mu_1 > \mu_2$, $\alpha < 0$ and $\alpha(\theta_0) > 0$. Then, the solution u^ϵ of Equation* (4.10) *satisfies*

$$\left\| \frac{du^\epsilon}{dx} \right\|_{L^2(\Omega)} \le C \|f_\epsilon\|_{L^2(\Omega)}, \quad \|u^\epsilon\|_{L^2(\Omega_1)} \le C\epsilon \|f_\epsilon\|_{L^2(\Omega)},$$

and

$$|u^\epsilon(0)| \le C\sqrt{\epsilon} \|f_\epsilon\|_{L^2(\Omega)}.$$

Assume that $\mu_1 > \mu_2$, $\alpha < 0$ and $\alpha(\theta_0) = 0$. Then, the solution u^ϵ of Equation (4.10) *satisfies*

$$\left\| \frac{du^\epsilon}{dx} \right\|_{L^2(\Omega_2)} \le C \|f_\epsilon\|_{L^2(\Omega)}, \quad \|u^\epsilon\|_{L^2(\Omega_1)} \le C\sqrt{\epsilon} \|f_\epsilon\|_{L^2(\Omega)},$$

and

$$\left\| e^{\theta_0 \frac{x}{\epsilon}} \frac{dv^\epsilon}{dx} \right\|_{L^2(\Omega_1)} \le C \|f_\epsilon\|_{L^2(\Omega)},$$

where $v^\epsilon = u^\epsilon \psi_1(x, x/\epsilon)/\psi_{1,\theta_0}(x, x/\epsilon)$ in Ω_1.

Proof of Theorem 3.11. Thanks to the a priori estimate of Lemma 6.2, the case $\mu_1 > \mu_2$, $\alpha < 0$ and $\alpha(\theta_0) > 0$ is completely similar to the case $\mu_1 > \mu_2$ and $\alpha \ge 0$, which has already been solved in Section 4.

Proof of Theorem 3.12. Let u^ϵ be the solution of (4.10) with right-hand side f_ϵ which is a bounded sequence in $L^2(\Omega)$. We introduce the function

$$\psi_{\theta_0}(x, y) = \chi_1(x)\psi_{1,\theta_0}(y) + \chi_2(x)\psi_2(y), \quad (6.2)$$

and define a new factorization (or change of unknown which is licit by virtue of Proposition 4.1),

$$v^\epsilon(x) = u^\epsilon(x) \frac{\psi\left(x, \frac{x}{\epsilon}\right)}{\psi_{\theta_0}\left(x, \frac{x}{\epsilon}\right)}.$$

Remark that $v^\epsilon = u^\epsilon$ in Ω_2, and $v^\epsilon(0) = u^\epsilon(0)$ (because of the normalization condition (2.2)). Testing variationally Equation (4.10) against $\frac{\psi_{\theta_0}(x,\frac{x}{\epsilon})}{\psi(x,\frac{x}{\epsilon})}\phi^\epsilon(x)$, where ϕ^ϵ is a test function in $H_0^1(\Omega)$, we obtain

$$\begin{aligned}&\int_\Omega D\left(x,\frac{x}{\epsilon}\right)\frac{du^\epsilon}{dx}\frac{d}{dx}\left(\frac{\psi_{\theta_0}\left(x,\frac{x}{\epsilon}\right)}{\psi\left(x,\frac{x}{\epsilon}\right)}\phi^\epsilon\right)dx \\ &\quad+\frac{\mu_1-\mu_2}{\epsilon^2}\int_{\Omega_1}B\left(x,\frac{x}{\epsilon}\right)u^\epsilon\left(\frac{\psi_{\theta_0}\left(x,\frac{x}{\epsilon}\right)}{\psi\left(x,\frac{x}{\epsilon}\right)}\phi^\epsilon\right)dx+\frac{1}{\epsilon}\alpha u^\epsilon(0)\phi^\epsilon(0) \\ &\quad=\int_\Omega f_\epsilon\left(\frac{\psi_{\theta_0}\left(x,\frac{x}{\epsilon}\right)}{\psi\left(x,\frac{x}{\epsilon}\right)}\phi^\epsilon\right)dx.\end{aligned} \tag{6.3}$$

Replacing u^ϵ by v^ϵ in its left-hand side, Identity (6.3) becomes

$$\begin{aligned}\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi_1^2\left(\frac{x}{\epsilon}\right)&\frac{d}{dx}\left(v^\epsilon\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}\right)\frac{d}{dx}\left(\phi^\epsilon\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}\right)dx \\ &+\int_{\Omega_2}D\left(x,\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx \\ &+\frac{\mu_1-\mu_2}{\epsilon^2}\int_{\Omega_1}\sigma_1\left(\frac{x}{\epsilon}\right)\psi_{1,\theta_0}^2\left(\frac{x}{\epsilon}\right)v^\epsilon\phi^\epsilon\,dx \\ &+\frac{1}{\epsilon}\alpha v^\epsilon(0)u^\epsilon(0) \\ &=\int_\Omega\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}f_\epsilon\phi^\epsilon\,dx.\end{aligned} \tag{6.4}$$

Note that

$$\begin{aligned}\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi_1^2\left(\frac{x}{\epsilon}\right)\frac{d}{dx}&\left(v^\epsilon\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}\right)\frac{d}{dx}\left(\phi^\epsilon\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}\right)dx \\ &=\int_{\Omega_1}a_1\left(\frac{x}{\epsilon}\right)\psi_{1,\theta_0}^2\left(\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx \\ &+\int_{\Omega_1}a_1\left(\frac{x}{\epsilon}\right)\frac{d}{dx}\left(\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\right)\frac{d}{dx}\left(v^\epsilon\phi^\epsilon\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\right) \\ &-\int_{\Omega_1}a_1\left(\frac{x}{\epsilon}\right)\frac{d}{dx}\left(\psi_1\left(\frac{x}{\epsilon}\right)\right)\frac{d}{dx}\left(\frac{\left(\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\right)^2}{\psi_1\left(\frac{x}{\epsilon}\right)}v^\epsilon\phi^\epsilon\right),\end{aligned}$$

and, by integration by parts and Definition (2.1) of $\psi_{1,\theta}$, we have

$$\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\frac{d}{dx}\left(\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\right)\frac{d}{dx}\left(v^\epsilon\phi^\epsilon\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\right)$$
$$-\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\frac{d}{dx}\left(\psi_1\left(\frac{x}{\epsilon}\right)\right)\frac{d}{dx}\left(\frac{\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}v^\epsilon\phi^\epsilon\right)$$
$$=\frac{1}{\epsilon}\left(\alpha(\theta_0)-\alpha\right)v^\epsilon(0)\phi^\epsilon(0)+\frac{\mu_2-\mu_1}{\epsilon^2}\int_{\Omega_1}\sigma_1\left(\frac{x}{\epsilon}\right)\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)v^\epsilon\phi^\epsilon.$$

As a consequence, Identity (6.4) becomes

$$\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx+\int_{\Omega_2} D\left(x,\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx \quad (6.5)$$
$$+\frac{1}{\epsilon}\alpha(\theta_0)v^\epsilon(0)\phi^\epsilon(0)=\int_\Omega \frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}f_\epsilon\phi^\epsilon\,dx.$$

Since $\alpha(\theta_0)=0$, we have

$$\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx+\int_{\Omega_2} D\left(x,\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx$$
$$=\int_\Omega \frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}f_\epsilon\phi^\epsilon\,dx.$$

Note that for any bounded sequence ϕ^ϵ in $W^{1,\infty}(\Omega)$,

$$\left|\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\frac{dv^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx\right|\le C\left\|e^{\theta_0\frac{x}{\epsilon}}\frac{dv^\epsilon}{dx}\right\|_{L^2(\Omega_1)}\left\|e^{\theta_0\frac{x}{\epsilon}}\right\|_{L^2(\Omega_1)}\to 0,$$

since $\|e^{\theta_0\frac{x}{\epsilon}}\frac{dv^\epsilon}{dx}\|_{L^2(\Omega_1)}$ is bounded, thanks to Lemma 6.2. Of course $\int_{\Omega_1}\frac{\psi_{1,\theta_0}\left(\frac{x}{\epsilon}\right)}{\psi_1\left(\frac{x}{\epsilon}\right)}f_\epsilon\phi^\epsilon$ goes to 0 exponentially fast. For such bounded ϕ^ϵ, (6.3) therefore, is written as

$$\int_{\Omega_2} D\left(x,\frac{x}{\epsilon}\right)\frac{du^\epsilon}{dx}\frac{d\phi^\epsilon}{dx}dx=\int_{\Omega_2} f_\epsilon\phi^\epsilon\,dx+o(1). \quad (6.6)$$

Since the test functions in the two-scale convergence method are of the type $\phi^\epsilon(x)=\phi^0(x)+\epsilon\phi^1(x,x/\epsilon)$ with smooth functions ϕ^0,ϕ^1, they are uniformly bounded in $W^{1,\infty}(\Omega)$ and one can use (6.6) to pass to the limit. Classical arguments of homogenization allow us to conclude the proof. $\square$

Proof of Lemma 6.2. With the choice $\phi^\epsilon=v^\epsilon$ in (6.5) we obtain

$$\int_{\Omega_1} a_1\left(\frac{x}{\epsilon}\right)\psi^2_{1,\theta_0}\left(\frac{x}{\epsilon}\right)\left(\frac{dv^\epsilon}{dx}\right)^2dx+\int_{\Omega_2} D\left(x,\frac{x}{\epsilon}\right)\left(\frac{dv^\epsilon}{dx}\right)^2dx \quad (6.7)$$
$$+\frac{1}{\epsilon}\alpha(\theta_0)\left(v^\epsilon\right)^2(0)$$
$$=\int_\Omega f_\epsilon u^\epsilon\,dx.$$

If $\alpha(\theta_0) > 0$, this implies that

$$|v^\epsilon(0)|^2 \le C\epsilon \, \|f_\epsilon\|_{L^2(\Omega)} \, \|u^\epsilon\|_{L^2(\Omega)} . \tag{6.8}$$

Because $u^\epsilon(0) = v^\epsilon(0)$, plugging (6.8) in (6.1) yields the desired results.

If $\alpha(\theta_0) = 0$, Identity (6.7) only implies that

$$\left\| e^{\theta_0 \frac{x}{\epsilon}} \frac{dv^\epsilon}{dx} \right\|^2_{L^2(\Omega_1)} \le C \, \|f_\epsilon\|_{L^2(\Omega)} \, \|u^\epsilon\|_{L^2(\Omega)} , \tag{6.9}$$

and

$$\left\| \frac{du^\epsilon}{dx} \right\|^2_{L^2(\Omega_2)} \le C \, \|f_\epsilon\|_{L^2(\Omega)} \, \|u^\epsilon\|_{L^2(\Omega)} .$$

We will next show that

$$\|u^\epsilon\|^2_{L^2(\Omega)} \le C \left(\epsilon \, \|f_\epsilon\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{L^2(\Omega_2)} \right) \tag{6.10}$$

and, together with (6.9) and Poincaré inequality in Ω_2, this yields the desired results.

Note that

$$u^\epsilon(0)^2 \le |\Omega_2|^2 \int_{\Omega_2} \left(\frac{du^\epsilon}{dx} \right)^2 \le C \, \|f_\epsilon\|_{L^2(\Omega)} \, \|u^\epsilon\|_{L^2(\Omega)} .$$

Using this inequality in (6.1) gives

$$\|u^\epsilon\|^2_{L^2(\Omega_1)} \le C\epsilon \, \|f_\epsilon\|_{L^2(\Omega)} \, \|u^\epsilon\|_{L^2(\Omega)} \le C\epsilon \left(\|f_\epsilon\|^2_{L^2(\Omega)} + \|u^\epsilon\|^2_{L^2(\Omega)} \right)$$

which in turn implies (6.10). □

7. Proof of Proposition 2.2

7.1. The case of C^2 coefficients

The first step is similar to the proof of Lemma 2.1, namely we transform (2.1) into (2.4). If we assume that the coefficients a_i, Σ_i and σ_i are C^2 periodic functions on [0, 1], then the first eigenfunction $\psi_{i,0}$ is actually differentiable two times, and thus the coefficients b and s of (2.4) are also of class C^2. Proposition 2.2 is then a consequence of Lemma 7.1.

Lemma 7.1. *Let b and s be periodic positive functions on* [0, 1] *such that their second derivative b'' and s'' exist and are piecewise continuous. Denote by $M > m > 0$ two positive constants which are the upper and lower bounds of b and s. For each $\theta \in \mathbb{R}$ the first eigenvector u_θ of Problem* (2.4) *with the normalization* $u_\theta(0) = 1$ *satisfies*

$$-C_1 - \frac{C_2}{\sqrt{-\mu(\theta)}} + C_3\sqrt{-\mu(\theta)} \le \frac{\theta}{|\theta|} u'_\theta(0) \le C_3\sqrt{-\mu(\theta)} + C_1 + \frac{C_2}{\sqrt{-\mu(\theta)}}, \tag{7.1}$$

where the positive constants C_1, C_2 and C_3 depend only on b and s.

Proof. The assumed smoothness of b and s enables us to perform a Liouville transformation of Problem (2.4). Introducing

$$t = \frac{1}{\gamma}\int_0^x \left(\frac{s(z)}{b(z)}\right)^{\frac{1}{2}} dz \, \gamma = \int_0^1 \left(\frac{s(z)}{b(z)}\right)^{\frac{1}{2}} dz, \quad \text{and} \quad f_\theta(t) = (s(x)b(x))^{\frac{1}{4}} u_\theta(x), \tag{7.2}$$

the transformed equation is, see [14],

$$\begin{cases} \dfrac{d^2 f_\theta}{dt^2}(t) + (\gamma^2 \mu(\theta) + Q(t)) f_\theta = 0 \text{ in } [0, 1], \\ t \to f_\theta(t)e^{-\theta t} \text{ 1-periodic,} \end{cases} \tag{7.3}$$

with

$$Q(t) = \gamma^2 b^{\frac{1}{4}}(x) s^{-\frac{3}{4}}(x) \frac{d}{dx}\left(b(x)\frac{d}{dx}\,(b(x)s(x))^{-\frac{1}{4}}\right).$$

We can assume without loss of generality that $\gamma = 1$. The boundary conditions are preserved since this change of variable preserves periodicity. We shall use the fact that Q is a bounded 1-periodic function. It is sufficient to prove (7.1) for $\theta > 0$, since in the other case the function $g_\theta(t) = f_\theta(-t)$ is solution of (7.3), with $\theta > 0$, if Q is replaced by $Q(-t)$, which is also a bounded 1-periodic function. By adding a constant to Q (and subtracting it from $\mu(\theta)$), we can always assume that $-M < Q(t) < -1$. On the other hand, thanks to Lemma 2.3, for sufficiently large θ we can also assume that $\mu(\theta)$ translated by the above constant is negative.

Next, we introduce g_1 and g_2 as the two fundamental solutions of the Cauchy problem for the ordinary differential equation $\frac{d^2 g}{dt^2} + (\mu(\theta) + Q(t))g = 0$, satisfying

$$g_1(0) = 1, \;\; g_1'(0) = 0, \text{and} g_2(0) = 0, \;\; g_2'(0) = 1.$$

It is a classical result of Floquet theory that $X_1 = e^\theta$ and $X_2 = e^{-\theta}$ are the roots of the characteristic equation

$$X^2 - \left(g_1(1) + g_2'(1)\right) X + 1 = 0.$$

By linearity, we can write $f_\theta(t) = f_\theta(0)g_1(t) + f_\theta'(0)g_2(t)$. Since $\theta > 0$, $e^\theta = \Delta + \left(\Delta^2 - 1\right)^{\frac{1}{2}}$ where $2\Delta = \left(g_1(1) + g_2'(1)\right)$. Consequently, $\Delta > 1$ and

$$g_1(1) + g_2'(1) = 2\Delta > e^\theta$$

and

$$e^\theta > 2\Delta - \frac{1}{\Delta} > 2\Delta - 2e^{-\theta} = g_1(1) + g_2'(1) - 2e^{-\theta}. \tag{7.4}$$

From the relation $f_\theta(1) = e^\theta f_\theta(0)$ we deduce that $\frac{f_\theta'(0)}{f_\theta(0)} g_2(1) = e^\theta - g_1(1)$. Using Relation (7.4) we have obtained

$$g_2'(1) > \frac{f_\theta'(0)}{f_\theta(0)} g_2(1) > g_2'(1) - 2e^{-\theta}. \tag{7.5}$$

Following Picard's iteration method (see e.g. [19]), we recursively define a sequence $(v_n(t))_{n\in\mathbb{N}}$ by

$$v_0(t) = \frac{1}{\omega}\sinh(\omega t)$$

and

$$v_n(t) = -\frac{1}{\omega}\int_0^t \sinh\left(\omega(t-\xi)\right) Q(\xi) v_{n-1}(\xi) d\xi \quad \text{for all } n \geq 1.$$

For $\omega = \sqrt{-\mu(\theta)}$, we find that $g_2(x) = \sum_{n=0}^{+\infty} v_n(x)$. Since $-\sinh\left(\omega(t-\xi)\right) Q(\xi) > 0$ for all $0 < \xi < t$, and $v_0(t) > 0$ for all $t > 0$, by induction, we can conclude that $W_n(x) \geq v_n(x) \geq w_n(x)$, for all $n \geq 0$ and $x \geq 0$, where W_n and w_n are two other sequences defined by $W_0 = v_0 = w_0$, and

$$W_n = \frac{M}{\omega}\int_0^t \sinh\left(\omega(t-\xi)\right) W_{n-1}(\xi) d\xi,$$
$$w_n = \frac{1}{\omega}\int_0^t \sinh\left(\omega(t-\xi)\right) w_{n-1}(\xi) d\xi \text{ for } n \geq 1.$$

Note that $W(t) = \sum_0^{+\infty} W_n(t)$ ($w(t) = \sum_0^{+\infty} w_n(t)$, respectively) is a solution of

$$\frac{d^2W}{dt^2} + (\mu - M)\, W = 0 \ \left(\frac{d^2w}{dt^2} + (\mu - 1)\, w = 0, \text{respectively}\right),$$

and therefore is given by $W(t) = \sinh\left(t\sqrt{M-\mu}\right)$ ($w(t) = \sinh\left(t\sqrt{1-\mu}\right)$, respectively) and consequently

$$\sinh\left(\sqrt{M-\mu}\right) = W(1) \geq g_2(1) \geq w(1) = \sinh\left(\sqrt{1-\mu}\right). \tag{7.6}$$

Similarly $W_n'(t) \geq v_n'(t) \geq w_n'(t)$, and

$$\sqrt{M-\mu}\cosh\left(\sqrt{M-\mu}\right) = W'(1) \geq g_2'(1) \geq w'(1) = \sqrt{1-\mu}\cosh\left(\sqrt{1-\mu}\right). \tag{7.7}$$

Using Inequalities (7.6) and (7.7) in (7.5) yields,

$$Ce^{\frac{M-1}{2\sqrt{-\mu}}}\sqrt{M-\mu} \geq \frac{f_\theta'(0)}{f_\theta(0)} \geq c\sqrt{1-\mu}e^{-\frac{M-1}{2\sqrt{-\mu}}} - 2e^{-\theta}.$$

Using the change of variables (7.2), and using the result of Lemma 2.3 to bound $e^{-\theta}$ in terms of $\mu(\theta)$ this inequality concludes the proof. □

7.2. The case of piecewise constant coefficients

As in the previous subsection, it is sufficient to consider System (2.4), which is equivalent to (2.1), and to study the case θ going to $+\infty$. As in Lemma 2.3, we rewrite (2.4) as a first-order system

$$\begin{cases} Y'(x) = A(x)Y(x) \\ Y(1) = e^{\theta} Y(0) \end{cases}, \quad A = \begin{bmatrix} 0 & b^{-1} \\ -\mu(\theta)s & 0 \end{bmatrix}, \text{ and } Y = \begin{pmatrix} Y_1 = u_\theta \\ Y_2 = bu'_\theta \end{pmatrix}.$$

Here we assume that the coefficients b, s are piecewise constant functions. More precisely, there exists a number N, a family of points $(x_i)_{0\le i\le N}$ satisfying $x_0 = 0 < x_{i-1} < x_i < x_{i+1} < x_N = 1$ for $2 \le i \le N-2$, and positive values $(b_i)_{1\le i\le N}$ and $(s_i)_{1\le i\le N}$ such that

$$b(x) = b_i \text{ and } s(x) = s_i \text{ for } x \in (x_{i-1}, x_i), \;\; 1 \le i \le N.$$

The goal is to prove that $Y_2(0)$ grows linearly as θ goes to $+\infty$, which in turn proves Proposition 2.2, since

$$\frac{d\psi_{i,\theta}}{dx}(0) = b^{-1}(0)Y_2(0) + \frac{d\psi_{i,0}}{dx}(0).$$

By Lemma 2.3 we already know that $\mu(\theta) < 0$ for $\theta \neq 0$ and has quadratic growth at infinity. A straightforward computation yields, for any $x \in (x_{i-1}, x_i)$,

$$\begin{aligned} Y(x) &= M_i(\theta, x)Y(x_{i-1}), \qquad (7.8)\\ M_i(\theta, x) &= \begin{bmatrix} \cosh\varphi_i(x) & \frac{1}{\sqrt{-\mu(\theta)b_i s_i}} \sinh\varphi_i(x) \\ \sqrt{-\mu(\theta)b_i s_i}\sinh\varphi_i(x) & \cosh\varphi_i(x) \end{bmatrix}, \end{aligned}$$

with $\varphi_i(x) = \sqrt{\frac{-\mu(\theta)s_i}{b_i}}(x - x_{i-1})$. Thus

$$Y(1) = M(\theta)Y(0) = e^{\theta}Y(0), \text{with} M(\theta) = \prod_{i=1}^{N} M_i(\theta, x_i).$$

Each matrix $M_i(\theta, x)$ has its determinant equal to 1, as well as $M(\theta)$. Thus the two eigenvalues of $M(\theta)$ are e^{θ} and $e^{-\theta}$. Let us compare these exact eigenvalues with those of the leading order term of $M(\theta)$ as θ goes to $+\infty$. Introducing $D(\theta) = \operatorname{diag}\left(\sqrt{-\mu(\theta)}, 1\right)$, we have

$$M_i(\theta, x_i) = e^{\varphi_i(x_i)} D(\theta)^{-1} M_i^0 D(\theta)\left(1 + O\left(e^{-\alpha\theta}\right)\right),$$

$$\text{with} \quad M_i^0 = \frac{1}{2}\begin{bmatrix} 1 & \frac{1}{\sqrt{b_i s_i}} \\ \sqrt{b_i s_i} & 1 \end{bmatrix},$$

and $\alpha = \min_{1\le i\le N}\left(2\,(x_i - x_{i-1})\sqrt{\frac{s_i m}{b_i M}}\right) > 0$. Therefore, noticing that $\sum_{i=1}^{N}\varphi_i(x_i) = C\sqrt{-\mu(\theta)}$, where $C > 0$ does not depend on θ, we obtain

$$M(\theta) = e^{C\sqrt{-\mu(\theta)}} D(\theta)^{-1} M^0 D(\theta)\left(1 + O\left(e^{-\alpha\theta}\right)\right), \text{ with } M^0 = \prod_{i=1}^{N} M_i^0.$$

Up to a small remainder, the eigenvalues of $M(\theta)$ are thus equal to those of M^0 times the multiplicative factor $\exp(C\sqrt{-\mu(\theta)})$. Since M^0 does not depend on θ, this proves that $\mu(\theta) = -c\theta^2 + o(1)$ for some positive constant $c > 0$. On the other hand, the eigenvectors of $M(\theta)$ are equal to $D(\theta)^{-1}$ times those of M^0 (up to a small remainder). Choosing the normalization $Y_1(0) = 1$, this yields that $Y_2(0) = c'\theta + o(1)$ for some constant c', which is positive as already remarked in the proof of Lemma 2.3.

Note added in proof. After submission of this paper for publication, we found an alternative proof of Lemma 5.4, which does not rely on Proposition 2.2. This enables us to prove Theorem 3.5 and Theorem 3.10 assuming only that the periodic coefficients are positive, bounded, measurable functions. This proof will be presented in a future work in collaboration with A. Piatnitski.

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