

A level-set method for shape optimization

Grégoire Allaire^a, François Jouve^a, Anca-Maria Toader^b

^a Centre de mathématiques appliquées, École polytechnique, 91128 Palaiseau, France

^b CMAF, Faculdade de Ciências da Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1699 Lisboa, Portugal

Received 5 April 2002; accepted 10 April 2002

Note presented by Philippe G. Ciarlet.

Abstract

We study a level-set method for numerical shape optimization of elastic structures. Our approach combines the level-set algorithm of Osher and Sethian with the classical shape gradient. Although this method is not specifically designed for topology optimization, it can easily handle topology changes for a very large class of objective functions. Its cost is moderate since the shape is captured on a fixed Eulerian mesh. *To cite this article: G. Allaire et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1125–1130.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Une méthode de lignes de niveaux pour l'optimisation de forme

Résumé

Nous proposons une méthode de lignes de niveaux pour l'optimisation de la forme de structures élastiques. Notre approche combine la méthode des lignes de niveaux d'Osher et Sethian et la dérivée classique de formes. Bien que cette méthode ne soit pas spécifiquement conçue pour faire de l'optimisation topologique, elle permet très facilement les changements de topologie de la forme d'une structure pour des fonctions objectifs très générales. Son coût en temps de calcul est modéré puisqu'il s'agit d'une méthode numérique de capture de formes sur un maillage eulérien fixe. *Pour citer cet article: G. Allaire et al., C. R. Acad. Sci. Paris, Ser. I 334 (2002) 1125–1130.* © 2002 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

L'optimisation de structures mécaniques est un domaine très important du point de vue des applications qui a connu récemment de nombreux progrès. À côté des méthodes classiques de variation de frontière (qui remontent au moins à Hadamard; voir par exemple [9,12,15,16]) est apparue une nouvelle méthode d'optimisation, dite topologique, basée sur la théorie de l'homogénéisation [1–5,8] et les références citées. Cette dernière méthode a un très faible coût de calcul car elle capture des formes sur un maillage fixe, mais elle est principalement restreinte à l'élasticité linéarisée. À la suite des travaux récents [10,14] nous proposons d'utiliser une méthode de lignes de niveaux pour faire de l'optimisation de formes en combinant, autant que faire se peut, les avantages des méthodes de variation de frontière et d'homogénéisation.

Suivant une idée de la méthode d'homogénéisation, nous utilisons un maillage fixe qui contient à la fois la forme et les trous (ou le vide) représentés par un matériau très faible. Le bord de la forme est paramétré

E-mail addresses: gregoire.allaire@polytechnique.fr (G. Allaire); francois.jouve@polytechnique.fr (F. Jouve); amtan@lmc.fc.ul.pt (A.-M. Toader).

par une fonction de ligne de niveaux suivant le formalisme d'Osher et Sethian [11,13]. L'optimisation de forme consiste à transporter la fonction ligne de niveaux (c'est-à-dire le bord de la forme) avec une vitesse qui fasse décroître la fonction objectif. Suivant la méthode de variation de frontière nous calculons cette vitesse en dérivant la fonction objectif par rapport à la forme.

Nous considérons un modèle mécanique d'élasticité linéarisée en deux ou trois dimensions d'espace et des fonctions objectifs régulières générales (la compliance ou un critère de moindres carrés ; voir (2) et (3)). Les tests numériques effectués montrent l'efficacité de notre algorithme en deux et trois dimensions.

Ce travail se distingue de l'étude précédente de Sethian et Wiegmann [14] car nous utilisons un gradient de forme et un matériau « mou » pour représenter le vide, ce qui nous permet de traiter des fonctions objectifs plus générales. Il se distingue aussi du travail d'Osher et Santosa [10] qui étudiait un problème de valeurs propres pour le Laplacien avec deux matériaux non dégénérés.

1. Introduction

Shape optimization of elastic structures is a very important and popular field. The classical method of shape sensitivity (or boundary variation) has been much studied (*see, e.g., [9,12,15,16]*). It is a very general method which can handle any type of objective functions and structural models, but it has two main drawbacks: its computational cost (because of remeshing) and its tendency to fall into local minima far away from global ones. The homogenization method (*see, e.g., [1–5,8]*) is an adequate remedy to these drawbacks but it is mainly restricted to linear elasticity and to particular objective functions (compliance, eigenfrequency, or compliant mechanism). Recently yet another method appeared in [10,14] based on the first approach of shape sensitivity but using the versatile level-set method for computational efficiency. The level-set method has been devised by Osher and Sethian [11,13] for numerically tracking fronts and free boundaries and it is used in many applications as motion by mean curvature, fluid mechanics, image processing, etc. In this paper we describe a new implementation of the level-set method for structural optimization.

2. Setting of the problem

To fix ideas we work in the linear elasticity setting, but there is no conceptual difficulty in choosing another model. Let $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) be a bounded open set occupied by a linear isotropic elastic material with Hooke's law A . For simplicity we assume that there is no volume forces but only surface loadings g . The boundary of Ω is made of three disjoint parts $\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D$, with Dirichlet boundary conditions on Γ_D , and Neumann boundary conditions on $\Gamma \cup \Gamma_N$. The boundary parts Γ_D and Γ_N are supposed to be fixed, and only Γ is allowed to vary in the optimization process.

The displacement field in Ω is the unique solution of the linearized elasticity system

$$\begin{cases} -\operatorname{div}(A e(u)) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ (A e(u))n = g & \text{on } \Gamma_N \cup \Gamma. \end{cases} \quad (1)$$

Since Γ is the variable of our optimization process, g must be known for all possible position of Γ . In most situations we shall have $g = 0$ on Γ , but g can possibly be non-zero on Γ in the case of an external pressure load exerted by a fluid.

The objective function is denoted by $J(\Omega)$. The most common choices for J are the compliance

$$J(\Omega) = \int_{\Gamma \cup \Gamma_N} g \cdot u \, ds = \int_{\Omega} A e(u) \cdot e(u) \, dx, \quad (2)$$

or a least square error compared to a target displacement

$$J(\Omega) = \left(\int_{\Omega} k(x)|u - u_0|^{\alpha} dx \right)^{1/\alpha}, \tag{3}$$

where $u = u(\Omega)$ is the solution of (1), u_0 is a given target displacement, $\alpha \geq 2$ and k is a given weighting factor. There are many other possible choices, some of which are under current study. To take into account the weight of the structure, we introduce a positive Lagrange multiplier ℓ and we minimize

$$\inf_{\Omega \text{ admissible}} \left(J(\Omega) + \ell \int_{\Omega} dx \right). \tag{4}$$

We shall not dwell on the precise definition of the set of admissible shapes. Let us simply mention that the existence theory for (4) is quite involved: it is well-known that, in general, (4) admits a minimizer only if some geometrical or topological restrictions on the shapes are enforced [6,7].

3. Shape derivative

In order to define a shape derivative we follow the approach of Murat and Simon [9,15] (see also [12, 16]). Starting from a reference domain Ω_0 , we consider domains of the type $\Omega = (\text{Id} + \tau)(\Omega_0)$, with $\tau \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ (for sufficiently small τ , $\text{Id} + \tau$ is a diffeomorphism). We further restrict the class of domains by asking that they all share the same parts of the boundary Γ_N and Γ_D : specifically, the map τ must belong to

$$T_{ad} = \{ \tau \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d) \text{ such that } \tau = 0 \text{ on } \Gamma_N \cup \Gamma_D \}.$$

The shape derivative of $J(\Omega)$ at Ω_0 is then defined as the Fréchet derivative in T_{ad} at 0 of the application $\tau \rightarrow J((\text{Id} + \tau)(\Omega_0))$. This notion is well defined and a standard computation shows that the shape derivative of (2) is

$$\left\langle \frac{\partial J}{\partial \Omega}(\Omega_0), \tau \right\rangle = \int_{\Gamma_0} \left(2 \left[\frac{\partial(g \cdot u)}{\partial n_0} + H_0 g \cdot u \right] - Ae(u) \cdot e(u) \right) \tau \cdot n_0 ds, \tag{5}$$

where Γ_0 is the variable part of the boundary of the reference domain Ω_0 , n_0 is the normal unit vector to Γ_0 , H_0 is the curvature of Γ_0 , and u is the solution of (1) in Ω_0 . Note that there is no adjoint state involved in (5) (indeed the minimization of (2) is a self-adjoint problem).

For the objective function (3) another computation yields the following shape derivative

$$\left\langle \frac{\partial J}{\partial \Omega}(\Omega_0), \tau \right\rangle = \int_{\Gamma_0} \left(\frac{\partial(g \cdot p)}{\partial n_0} + H_0 g \cdot p - Ae(p) \cdot e(u) + \frac{C_0}{\alpha} k|u - u_0|^{\alpha} \right) \tau \cdot n_0 ds, \tag{6}$$

where u is the solution of (1) in Ω_0 , and p is the adjoint state in Ω_0 . The adjoint problem is defined by

$$\begin{cases} -\text{div}(Ae(p)) = C_0 k(x)|u - u_0|^{\alpha-2}(u - u_0) & \text{in } \Omega, \\ p = 0 & \text{on } \Gamma_D, \\ (Ae(p))n = 0 & \text{on } \Gamma_N \cup \Gamma, \end{cases}$$

where C_0 is a constant given by $C_0 = \left(\int_{\Omega} k(x)|u(x) - u_0(x)|^{\alpha} dx \right)^{1/\alpha-1}$. As is well known, the shape derivatives (5) and (6) depend only on the normal trace of τ on the boundary Γ .

4. Numerical algorithm

We apply a gradient method to the minimization problem (4), using the shape derivatives of Section 3 to compute the numerical gradient (not forgetting to add the contribution of the weight of the structure).

For numerical purposes we introduce a working domain $Q \subset \mathbb{R}^d$ in which all admissible shapes Ω are included, i.e., $\Omega \subset Q$, and we assume that Γ_D and Γ_N are parts of ∂Q . In order to describe the boundary of Ω we introduce a level-set function ψ defined on Q such that

$$\begin{cases} \psi(x) = 0 \Leftrightarrow x \in \partial\Omega \cap Q, \\ \psi(x) < 0 \Leftrightarrow x \in \Omega, \\ \psi(x) > 0 \Leftrightarrow x \in (Q \setminus \Omega). \end{cases}$$

The normal n to the shape Ω is recovered as $\nabla\psi/|\nabla\psi|$ and the curvature H is given by the divergence of n (these quantities are evaluated by finite differences since our mesh is uniformly rectangular). Note that, although n and H are defined on Γ , the level set method allows to define easily their extension in the whole domain Q . We fill the void part $Q \setminus \Omega$ with a very weak material with Hooke's law $B = 10^{-3}A$ and we perform the elasticity analysis on a fixed rectangular mesh in Q (using $Q1$ finite elements). Since n and H , as well as u and p , are computed everywhere in Q , formulas (5) and (6) deliver a vector field V throughout the domain and not only on the free boundary Γ . For example, (5) yields

$$V = vn \quad \text{with } v = 2 \left[\frac{\partial(g \cdot u)}{\partial n} + Hg \cdot u \right] - Ae(u) \cdot e(u).$$

After evaluating the gradient of $J(\Omega)$, or equivalently this vector field V , we transport the level set function ψ along this gradient flow $-V = -vn$. Since $n = \nabla\psi/|\nabla\psi|$, we end up with the following Hamilton–Jacobi equation

$$\frac{\partial\psi}{\partial t} - v|\nabla\psi| = 0, \tag{7}$$

where the time variable t plays the role of the descent step in the gradient algorithm. Transporting ψ by (7) is equivalent to move the boundary of Ω (the zero level set of ψ) along the descent gradient direction $-\partial J/\partial\Omega$. We solve (7) using a standard explicit upwind finite difference scheme (see, e.g., [13]). Finally, our algorithm is an iterative method, structured as follows:

- (1) Initialization of the level-set function ψ_0 corresponding to an initial guess Ω_0 .
- (2) Iteration until convergence, for $k \geq 0$:
 - (a) Computation of u_k and p_k through two problems of linear elasticity in Q with Hooke's law $A_k(x) = A$ where $\psi_k(x) < 0$ and $A_k(x) = 10^{-3}A$ where $\psi_k(x) > 0$.
 - (b) Deformation of the shape through the transport of the level set function: $\psi_{k+1}(x) = \psi(\Delta t_k, x)$ where $\psi(t, x)$ is the solution of (7) with velocity $v_k = v(u_k, p_k)$ and initial condition $\psi(0, x) = \psi_k(x)$. The time step Δt_k is chosen such that $J(\Omega_{k+1}) + \ell|\Omega_{k+1}| \leq J(\Omega_k) + \ell|\Omega_k|$.

This algorithm never creates new holes or boundaries if the time step Δt_k satisfies a CFL condition for (7) (there is no nucleation mechanism for new holes). However the level set method is well known to handle easily topology changes, i.e., merging or cancellation of holes. Therefore, our algorithm is able to perform topology optimization if the number of holes of the initial design is sufficiently large (see Fig. 1). The algorithm converges smoothly to a (local) minimum which strongly depends, of course, on the initial topology (see the differences in Figs. 2 and 3). For a “good” initialization, the numerical results are very similar to those obtained by the homogenization method but the convergence is usually slower (although we did not yet try to speed it by a quasi-Newton algorithm). From time to time, for stability reasons, we also reinitialize the level set function ψ in order that it be the signed distance function to the boundary of the current shape Ω (see [13]).

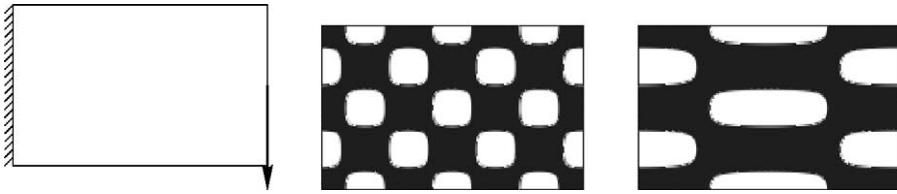


Figure 1. – Boundary conditions and two initializations of a 2-d cantilever.

Figure 1. – Conditions aux limites et deux initialisations d'une console 2-d.



Figure 2. – Two-dimensional cantilever with the initialization of Fig. 1 (middle).

Figure 2. – Console bi-dimensionnelle avec l'initialisation de la Fig. 1 (centre).



Figure 3. – Two-dimensional cantilever with the initialization of Fig. 1 (right).

Figure 3. – Console bi-dimensionnelle avec l'initialisation de la Fig. 1 (droite).

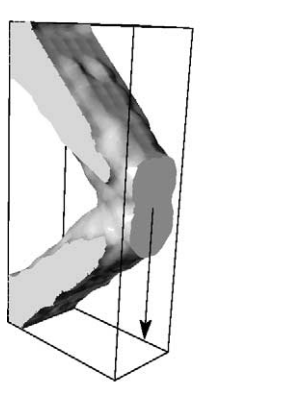


Figure 4. – Three-dimensional short cantilever.

Figure 4. – Console courte en trois dimensions.

We give some numerical results for the compliance objective function (2) with $g = 0$ on Γ (no design dependent loads). The boundary conditions and two initial configurations for a plane cantilever

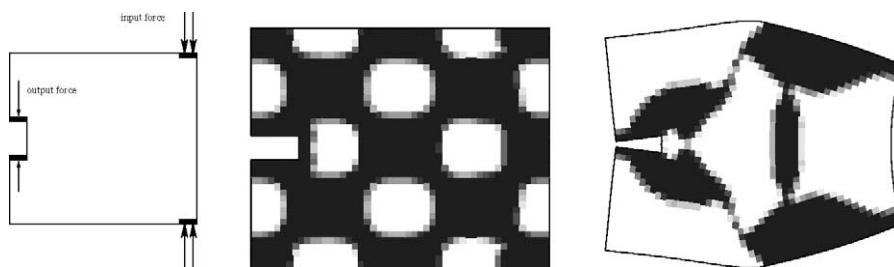


Figure 5. – Boundary conditions, initialization, and deformed optimal shape of a plane gripping mechanism.

Figure 5. – Condition aux limites, initialisation et forme optimale déformée d'un mécanisme plan pour pincer.

are displayed on Fig. 1. The results are shown on Figs. 2 and 3 for an increasing number of iterations. A three-dimensional short cantilever is given on Fig. 4.

Next, Fig. 5 shows a numerical result for the least square objective function (3) with $g = 0$ on Γ . This is a classical gripping mechanism test case which is described, e.g., in [1].

Our method is different from that in [14] for at least two reasons. First, we do not use the immersed interface method to compute the elastic fields, but rather the simpler “ersatz material” approach which amounts to fill the holes by a weak phase. Second, we use an exact continuous gradient formula for the transport of the level set function instead of an ad hoc criteria based on the von Mises equivalent stress.

Acknowledgements. The work of A.-M. Toader at Ecole Polytechnique has been supported by the grant SFRH/BPD/3591/2000 of the Fundação para a Ciência e a Tecnologia, Portugal.

References

- [1] G. Allaire, *Shape Optimization by the Homogenization Method*, Springer-Verlag, New York, 2001.
- [2] G. Allaire, E. Bonnetier, G. Francfort, F. Jouve, *Shape optimization by the homogenization method*, *Numer. Math.* 76 (1997) 27–68.
- [3] G. Allaire, R.V. Kohn, *Optimal design for minimum weight and compliance in plane stress using extremal microstructures*, *European J. Mech. A Solids* 12 (6) (1993) 839–878.
- [4] M. Bendsoe, *Methods for Optimization of Structural Topology, Shape and Material*, Springer-Verlag, New York, 1995.
- [5] M. Bendsoe, N. Kikuchi, *Generating optimal topologies in structural design using a homogenization method*, *Comput. Methods Appl. Mech. Engrg.* 71 (1988) 197–224.
- [6] A. Chambolle, *A density result in two-dimensional linearized elasticity and applications*, Preprint CEREMADE 121, Université Paris-Dauphine, 2001.
- [7] D. Chenaïs, *On the existence of a solution in a domain identification problem*, *J. Math. Anal. Appl.* 52 (1975) 189–289.
- [8] A. Cherkaev, *Variational Methods for Structural Optimization*, Springer-Verlag, New York, 2000.
- [9] F. Murat, S. Simon, *Études de problèmes d’optimal design*, in: *Lecture Notes in Comput. Sci.*, Vol. 41, Springer-Verlag, Berlin, 1976, pp. 54–62.
- [10] S. Osher, F. Santosa, *Level set methods for optimization problems involving geometry and constraints: frequencies of a two-density inhomogeneous drum*, *J. Comput. Phys.* 171 (2001) 272–288.
- [11] S. Osher, J.A. Sethian, *Front propagating with curvature dependent speed: algorithms based on Hamilton–Jacobi formulations*, *J. Comput. Phys.* 78 (1988) 12–49.
- [12] O. Pironneau, *Optimal Shape Design for Elliptic Systems*, Springer-Verlag, New York, 1984.
- [13] J.A. Sethian, *Level Set Methods and Fast Marching Methods: Evolving Interfaces in Computational Geometry, Fluid Mechanics, Computer Vision and Materials Science*, 1999.
- [14] J. Sethian, A. Wiegmann, *Structural boundary design via level set and immersed interface methods*, *J. Comput. Phys.* 163 (2000) 489–528.
- [15] J. Simon, *Differentiation with respect to the domain in boundary value problems*, *Numer. Funct. Anal. Optimiz.* 2 (1980) 649–687.
- [16] J. Sokolowski, J.P. Zolesio, *Introduction to Shape Optimization: Shape Sensitivity Analysis*, Springer Ser. Comput. Math., Vol. 10, Springer, Berlin, 1992.